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PROBLEMS OF THE SYNTHESIS OF RADAR SIGNALS. (U)

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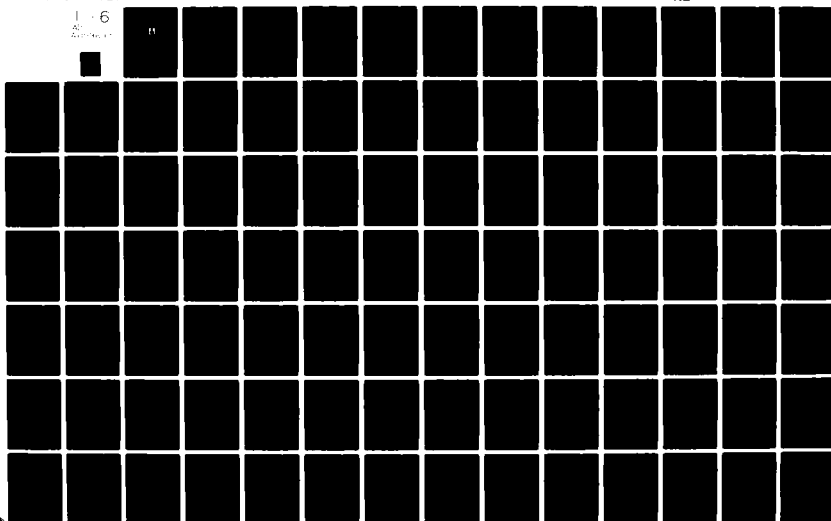
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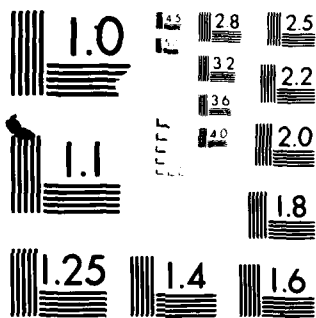
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PROBLEMS OF THE SYNTHESIS OF RADAR SIGNALS

by

D. Ye. Vakman, R. M. Sedletskiy



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# U. S. BOARD ON GEOGRAPHIC NAMES transliteration SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	<i>А а</i>	A, a	Р р	<i>Р р</i>	R, r
Б б	<i>Б б</i>	B, b	С с	<i>С с</i>	S, s
В в	<i>В в</i>	V, v	Т т	<i>Т т</i>	T, t
Г г	<i>Г г</i>	G, g	У у	<i>У у</i>	U, u
Д д	<i>Д д</i>	D, d	Ф ф	<i>Ф ф</i>	F, f
Е е	<i>Е е</i>	Ye, ye; E, e*	Х х	<i>Х х</i>	Kh, kh
Ж ж	<i>Ж ж</i>	Zh, zh	Ц ц	<i>Ц ц</i>	Ts, ts
З з	<i>З з</i>	Z, z	Ч ч	<i>Ч ч</i>	Ch, ch
И и	<i>И и</i>	I, i	Ш ш	<i>Ш ш</i>	Sh, sh
Й й	<i>Й й</i>	Y, y	Щ щ	<i>Щ щ</i>	Shch, snch
К к	<i>К к</i>	K, k	Ъ ъ	<i>Ъ ъ</i>	"
Л л	<i>Л л</i>	L, l	Ы ы	<i>Ы ы</i>	Y, y
М м	<i>М м</i>	M, m	Ь ь	<i>Ь ь</i>	'
Н н	<i>Н н</i>	N, n	Э э	<i>Э э</i>	E, e
О о	<i>О о</i>	O, o	Ю ю	<i>Ю ю</i>	Yu, yu
П п	<i>П п</i>	P, p	Я я	<i>Я я</i>	Ya, ya

\*ye initially, after vowels, and after ъ, ы; e elsewhere.  
When written as ё in Russian, transliterate as yě or ě.

## RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	sinh
cos	cos	ch	cosh	arc ch	cosh
tg	tan	th	tanh	arc th	tanh
ctg	cot	cth	coth	arc cth	coth
sec	sec	sch	sech	arc sch	sech
cosec	csc	csch	csch	arc csch	csch

### Russian English

rot	curl
lg	log

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PAGE 1

Page 1.

PROBLEMS OF THE SYNTHESIS OF RADAR SIGNALS.

D. Ye. Vakman, R. M. Sedletskiy.

Page 2.

Is examined the wide circle of the tasks, connected with the optimization of the signals, used in the radar. All tasks are treated from the positions of the criterion of proximity - new universal approach to the synthesis, applied not only in the theory of signals, but also in other regions. It is shown that the synthesis of signals according to the functions the uncertainties/indeterminancies and according to the autocorrelation functions, and also the optimization of most commonly used signals with the frequency modulation and with the phase manipulation can successfully be carried out on the basis of the general/common/total approach indicated. Are developed/processed also the iterative methods of synthesis with the use/application of the nonclassical calculus of variations. Are given the new results, obtained by the authors.

The book is intended for scientific workers, graduate students and engineers, who are interested general by questions of radar and theory of signals, and also in problems of synthesis in other regions.

Page 3.

PREFACE.

The characteristic feature of contemporary radio electronics is the wide use of serrated signals, i.e., signals whose product of duration to the width of the spectrum considerably exceeds unity. As the confirmation of the aforesaid can serve the following data about a quantity of patents, given cut in the series/row of the foreign countries (USA, Great Britain, FRG, France) according to the methods of formation, processing and on the uses/applications of the complicated sounding signals in radar [41]:

(1) Годы	(2) Выдано патентов
1961	4
1962	18
1963	36
1964	55
1965	78
1966	88
1967	34

Key: (1). Years. (2). Patents issued.

For the years 1961-1967 313 patents were issued.

The use of serrated signals is connected not only with the solution of serious technical problems (about which testifies the

mentioned flow of inventions), but it requires also in-depth theoretical studies, especially on the synthesis, the optimization of the structure of signals themselves. This question already has vast literature. Of the books, published in the Russian language, it is possible to mention the monograph of Varakin [13], Slcka [62], Petrovich and Razmakhnin [48], Ccek and Bernfeld [35], and also work of one of the authors of this book [7, 8].

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Should be noted also the excellent book of Franks on the principles of the theory of signals [30]. All these books touch on one way or another questions of synthesis.

But the known methods of synthesis are very heterogeneous. Depending on the structure of signal and concrete/specific/actual requirements are applied the different methods of the solution, different criteria of approximation/approach and so forth, etc. Thus, the methods of the synthesis of the signals with the frequency modulation in practice do not have general/common/total with those which usually are used for the discrete/digital signals with the phase manipulation.

Meanwhile to the diverse tasks of the synthesis of signals, and

also synthesis of antennas, filters and other units, is characteristic certain generality, which makes it possible to formulate these tasks from unity of opinion. This universal approach simplifies the understanding of different problems of synthesis and, as frequently it is during similar generalizations, offers further possibilities in the solution of the tasks, almost inaccessible for the methods, which were being applied earlier. The development/detection of such general/common/total approach is the fundamental purpose of this work.

For this is used the representation of signals (or other objects of synthesis) in the form of multidimensional vectors in certain abstract space. This representation is widely known from the theory of freedom from interference and adjacent regions, it is based on the simplest positions of functional analysis. In application to the problem of synthesis this representation makes it possible to obtain the demonstrative geometric description of the corresponding tasks. It is clarified, that independent of the nature of the objects of synthesis and concrete/specific/actual requirements the problem is reduced to the minimization of the distance between some sets in the appropriate space. This position, named the hypothesis (or criterion) of proximity, is the basis of this work.

This hypothesis was formulated by one of the authors in 1967

[8]. This book is, thus, by the development of the work indicated, moreover substantially are used and are generalized authors' previous publications [7-12, 53, 60, 63].

One should emphasize that the approach in question to the synthesis contains only the deterministic tasks when the desired property of synthesized objects is formulated without the use/application of statistical criteria of optimum character. Such tasks meet very frequently.

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They include the synthesis of antennas according to the radiation patterns, the synthesis of filters according to the frequency characteristics, the synthesis of signals according to the functions of uncertainty/indeterminacy, etc. But besides the mentioned deterministic treatment for the synthesis of signals are frequently used the probabilistic criteria, similar by that used in the theory of detection and evaluation/estimate of the parameters. In this connection to material of the book is presupposed the input chapter, which elucidates role and place of the deterministic methods of the synthesis of signals in the series/row of others.

In chapter 1 is in detail presented the proposed universal

approach to the synthesis, while in the subsequent chapters it is applied to some tasks in the optimization of signals. In this case into some cases we come only to the new treatment of known results, in others - are achieved the generalizations, which have independent value; finally there is a series of problems where this approach leads to the new results.

Some of the examined in the book tasks have the general/common/total value in the given region, others are of interest because are determined the signals with good in this or another sense properties. But it goes without saying the book does not claim to the complete scope/coverage of the problem of the synthesis of signals. The authors attempted, mainly, to consider different tasks, using single method, of confirming the universality of the latter.

Method is adapted for the synthesis of single (in particular, that sound) signals, the problem of the synthesis of the groups of signals according to the mutual-correlation properties in the book is not examined. In the equal measure are not examined any questions, connected with the construction of systems as a whole.

The synthesis of signals as other tasks of optimization, is reduced to the variation problems. The known methods of solving th

variational problems can be divided into two large groups. Classical calculus of variations gives the analytical resolutions of series of problems or, at least, reduces them to other problems of analysis, to the differential or integral equations. But today intensely they are applied and are just as intensely developed also nonclassical variational methods, based on the iterations, the successive approximations to the unknown solution.

Page 6.

Contemporary computer technology makes it possible to apply iterative numerical methods with the great success, and it is possible to hear propositions about the fact that the analytical methods became obsolete, they are less efficient in the practical tasks than numerical, iterative.

Hardly it is possible with this to agree. Classical and nonclassical variational methods mutually supplement each other; in the complex problems of the synthesis of signals it is expedient to join those, etc. The authors of this book hope that their collaboration contributed to this interpenetration of methods.

Introduction, chapter 1-3, 6-9 (besides §9.11) are written by D. Ye. Vakman, remaining sections of book - together by both authors. By

R. M. Sedletskiy are performed also calculations on TsVM [digital computer] for obtaining the concrete/specific/actual results.

The authors are grateful to the doctor of technical sciences L. Ye. Varakin and to the doctor of technical sciences A. M. Trakhtman for the critical observations and the councils, which contributed to an improvement in the book.

Page 7.

## INTRODUCTION.

### PROBLEM OF THE SYNTHESIS OF SIGNALS IN A RADAR.

Short historical survey/coverage.

Examining questions of use/application in the radar of the sounding signals of different structures, it is possible to isolate several historical stages. In the first development period a question about the selection of waveform, in fact, was not placed. The practical possibilities of generation and processing of signals were so limited that were applied or the single-frequency impulses/moments/pulses of the short duration, close to the continuous ones, obtained from usual type vacuum-tube oscillators. Respectively even during the first stage were demarcated two directions - pulse and continuous. To each of them were characteristic their limitations, each solved its problems. Pulse method was applied in the devices/equipment with a comparatively long range of action, continuous - with the low, but in this case was reached higher accuracy.

The following, second stage, is connected with the advent of a pulse-coherent technology for the selection of moving objects. Selection is reached due to the improved methods of processing the echo signals on the condition that an indispensable sequence of sounding pulses possesses sufficiently high stability. They are required, in particular, the low frequency drifts of filling for several periods. In a certain sense such a requirement draws together pulse radar with the continuous.

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At least characteristic time scale increases several the orders: it is not the duration of single pulse, but repetition period. Coherent pulse sequence is already the serrated signal, whose product of duration to the width of the spectrum is sufficiently great.

But this understanding arrived somewhat later, in third, contemporary development stage of radar technology. Certain threshold of this stage it is possible to consider the first successes of the statistical theory of radar, which relate to the middle Fifties when it was established/installed, in particular, that the most important characteristic of RLS (probability of detection) is determined with the optimum reception/procedure by energy of the sounding signal, but do not depend on the special features/peculiarities of its form,

including from the pulse power, durations, widths of the spectrum and so forth, etc. This it indicated the possibility to vary waveform, satisfying other requirements with the maximum detectable range.

The straight/direct continuation of this idea is the widely utilized today technique of the compression of impulses/momenta/pulses, which makes it possible to raise accuracy and range resolution at the limited peak power and long range of detection. It is possible to say that compression technique joins somehow the advantages of pulse and continuous methods in the radar. Used for this sounding signals possess the wide spectrum for the large duration, these are serrated signals with the frequency modulation, the phase manipulation and the like<sup>1</sup>.

FOOTNOTE <sup>1</sup>. The first publication on the use/application of the complicated sounding signals in the radar pertains to the year 1960 [39]. From the Soviet sources should be mentioned Ya. D. Shirman's invention [86, 87], who in 1956 proposed analogous method. In 10-15 years of development the compression technique of serrated signals achieved surprising successes. Is today realized compression of ChM signal  $10^6$  times (!), the duration of the sounding signal being 1 ms, and the duration of compressed - only of 1 ns. By this is provided resolution on the order of 15 cm [37]. ENDFOOTNOTE.

The middle Fifties includes general/common/total posing of the question about the simultaneous measurements of all parameters of the motion of object - its coordinates, rate and acceleration. This also was done in the plan/layout of the statistical treatment of problem, but made it possible to determine the effect of waveform on the quality of the measurements indicated.

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Here should be noted Woodward's basic work [16], who for the first time introduced the generalized characteristic of the sounding signal - function of uncertainty/indeterminacy, which comparatively fully describes the effect of the latter on the measurements of the delay time and frequency. The function of uncertainty/indeterminacy is similar to the antenna radiation pattern: in the same measure in which the radiation pattern characterizes accuracy and resolution of angular measurements, the function of uncertainty/indeterminacy characterizes accuracy and resolution of rangings and rate.

Logically, as soon as was understood this value of the function of uncertainty/indeterminacy, was begun the detailed study of its properties and were done the first attempts at the synthesis of the signals, which possess the desired functions of uncertainty/indeterminacy.

Beginning from the end/lead Fifties, this complicated in mathematical sense problem of synthesis intensely is developed/processed and abroad. To some methods of its solution, in the opinion of the authors sufficient to efficient ones, and is dedicated this book.

In parallel with the development of the theory of serrated signals occurred the development of technology of their generation. Technical capabilities considerably were widened in the latter/last decade, and although from the point of theorist's view these possibilities still leave to desire the best, the contemporary stage of radar it is unconditionally characterized by the wide application of diverse serrated signals. Therefore the methods of synthesis, optimization of these signals, especially those of them, which to the maximum degree consider the possibilities of generation, do not lose and cannot lose urgency.

Most recently was planned, apparently, new, fourth stage in the process of the improvement of the structure of signals. This stage is connected with the general/common/total tendency of the use/application of the self-tuning, adaptive devices/equipment in the radar. In the initial setting this question raises even to the

sequential analysis of Wald. But if at the early stages the technical capabilities of self-adjusting were limited in essence to variable speed of scanning on the angle, then today there is a possibility in principle to change also the structure of signal and the method of reception/procedure in the dependence on the observed situation, according to the previous observations.

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As a characteristic example of this type can serve the radar system, which, after determining roughly location and target speed, and having also considered interference situation, automatically changes the form of the sounding signal and (or) the method of reception/procedure in order to in the best way isolate signal from the predicted target from the available interferences, and to also make more precise its coordinates. In proportion to the refinement of real situation this process of adaptation of RIS continuously is continued<sup>1</sup>.

FOOTNOTE <sup>1</sup>. In the literature already there are indications about the development of similar adaptive RIS, which automatically change the operating mode in the dependence on the results of previous observations [54]. ENDFOOTNOTE.

The synthesis of signals for the adaptive systems is characterized by the series/row of special features/peculiarities. The selection of signal, its optimization, must be produced not at the writing desk and not after the panel of TSVM, when for obtaining the results and for their analysis can be expended/consumed hours or week, but it is direct in the equipment, during several seconds or milliseconds. Of course so rigid a regulation of time sets substantial limitations on the methods of synthesis. But nevertheless more fundamental is another special feature/peculiarity, connected with the fact that only the adaptive systems make it possible to virtually obtain necessary, a priori for the task of synthesis, information about the concrete/specific/actual situation. This special feature/peculiarity in detail is considered below.

Operational description of locating system.

Let at the point of space, characterized by radius-vector  $r$  (origin of coordinates is combined with RLS), there is a pinpoint target, which moves at a rate of  $v$ . If the sounding signal is  $s(t)$ , then with the usual assumptions about not too high a target speed and about the narrow-band characteristic of signal (with respect to the carrier frequency) the echo signal can be presented in the form

$$x_e(t) = \frac{a(r)}{r^2} g^2(r) s(t - \tau) e^{j\omega \tau}.$$

Here  $r = |r|$  - the range of target,  $g(r)$  - the antenna radiation pattern

(on the field),  $\tau=2r/c$  - delay of the echo signal,  $\Omega=2\omega_0 v_r/c$  - the Doppler shift of the carrier frequency  $\omega_0$ ,  $v_r$  - the radial velocity of target.

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The composite coefficient of reflection  $a(r)$  characterizes the level of the echo signal (reflecting surface) and the phase of reflection.

In general target is not point, it occupies certain space  $V_0$  in the space, and then

$$x_a(t) = \int_{V_0} \frac{a(r, t)}{r^2} g^2(r) s(t - \tau) e^{j\Omega t} dV. \quad (1a)$$

Moreover here  $a(r, t)$  is density of reflection coefficients, so that  $a(r, t) dV$  is the coefficient of reflection of volume element. Dependence on  $t$  characterizes changes (usually slow) in coefficient of reflection, for example, due to the motion of target.

Besides useful signal  $x_a(t)$  on FLS come mixing reflections  $x_b(t)$  from other reflecting objects. These objects are characterized by density  $b(r, t)$ , by analogous  $a(r, t)$ , and they occupy certain region  $V_0$  in the space. It is analogous with previous

$$x_b(t) = \int_{V_0} \frac{b(r, t)}{r^2} g^2(r) s(t - \tau) e^{j\Omega t} dV. \quad (1b)$$

Finally, the input of receiver enter noises and other additive interferences. The sources of such interferences also are somehow distributed in the space, but it goes without saying interference level does not depend on the form of the sounding signal, but radiation pattern here participates only in the process of reception/procedure. Therefore, designating noise component of the signal through  $x_n(t)$ , it is possible to register

$$x_n(t) = \int_{V_n} \frac{n(r, t)}{r} g(r) dV, \quad (18)$$

where  $n(r, t)$  - the density of the sources of additive interferences, and  $V_n$  - counterpart of the space.

The resulting input signal is put together of three that indicated the component

$$x(t) = x_a(t) + x_r(t) + x_n(t).$$

These components are random functions, since they depend, for example, on the random location of the reflecting objects.

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We will thus far assume the statistical description of the observed situation known, i.e., consider known probabilistic distributions for values  $a(r, t)$ ,  $b(r, t)$  and  $n(r, t)$ . Input signal depends also on characteristics of RLS - form of the sounding signal  $s(t)$  and

radiation pattern  $g(r)$ . The problem of synthesis (design) lies in the fact that to select these characteristics by certain in better shape.

Receiver of RLS always contains linear or quasi-linear input part - the circuit of amplification on the high and in the intermediate frequency. For the brevity we will call this part simply receiver. This receiver can be described by its pulse reaction  $h(t)$ , the output signal of receiver exists a roll of input and  $h(t)$ :

$$y(t) = \int_{-\infty}^{+\infty} x(t') h(t-t') dt'. \quad (II)$$

Together with  $s(t)$  and  $g(t)$  receiver response  $h(t)$  also must be synthesized on without some conditions of optimum character.

From previous it is not difficult to comprehend that RLS can be treated as certain operator, which converts the characteristics of the observed situation into the output signal of receiver  $y(t)$ . Grouping the functional arguments of this operator which depend on structure of RLS and are subject to optimization, and the arguments, which depend only on the objects of observation, can be registered

$$y = Y(s, g, h; a, b, n). \quad (III a)$$

Operator  $Y$  has comparatively simple structure, he is linear relative to all his arguments, except radiation pattern (where the dependence is quadratic). In particular, the output signal  $y(t)$  is put together of three components, connected with  $a(r, t)$ ,  $b(r, t)$  and

$n(r, t)$  respectively, so that occurs the superposition

$$y(t) = y_o(t) + y_b(t) + y_n(t).$$

In the operational designations

$$Y = Y_o(s, g, h) + Y_b(s, g, h) + Y_n(s, g, h). \quad (III 6)$$

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Further, because of the linearity of the conversion, realized by a receiver,

$$Y = HX_o(s, g) + HX_b(s, g) + HX_n(s, g). \quad (III B)$$

where operators  $H$  and  $X$  are determined concretely/specifically/actually by relationships/ratios (II) and (I).

The output signal  $y(t)$  is used for determining the parameters of motion or other characteristics of targets. For example, can be monitored the range of target, its angular position, rate, and also number of targets, their reflecting surfaces, etc. Let us designate the controllable/controlled/inspected parameters through  $z_1, z_2, \dots$ , and their set - by multidimensional vector  $z$ .

According to locating observations is determined the in general not true beam vector of parameters  $z$ , but its only certain approximate estimate  $\hat{z}$ . Evaluation/estimate is formed as a result

of some actions, produced above the output signal of receiver  $y(t)$  - by the only source of information about the observed objects. These actions are implemented automatically or with the participation of man, but it is possible to assume that for each system is certain regular algorithm (instruction for the operator), permitting to obtain the vector of evaluations/estimates  $z$  according to the realization of the output signal accepted:

$$\hat{z} = Z[y(t)]. \quad (IV)$$

Operators  $Y$  and  $Z$  give from a fundamental point of view the complete description of locating system. Operator  $Y$  characterizes the formation of the echo signals in "ether/ester" and their conversion in the receiver, he considers also the form of the sounding signal and the antenna radiation pattern. This operator, identical for all radar systems, is defined concretely/specifically/actually with the help of the previous relationships/ratios and, as it seemed, it was linear relative to almost all its arguments.

The second operator  $Z$  characterizes further processing of the echo signals, beginning from the detection. The concrete/specific/actual structure of this operator is more complicated, it depends on many factors and, in the first place, from equipment usage.

Depending on type of RLS are monitored one or the other parameters of targets - components of vector  $z$ . Into a number of tasks of treatment can enter in general such "global" operations, as the evaluation/estimate of situation as a whole or determination of the type of the targets, which relate to pattern recognition. Even such cases can be included/connected in our description, after assuming that operator  $Z$  maps many signals  $y(t)$  to the discrete set of the possible solutions about the situation. The simplest version of this type is a detection problem when vector  $z$  allows/assumes only the two values: 1 (target of eating) or 0 (there is no target).

Straight/direct and indirect approaches to the synthesis.

Let us attempt to formulate the task of the synthesis of signals and other characteristics of locating system in a strict form.

The vector of parameters  $z$ , understood in the generalized, indicated above sense, completely characterizes the designation/purpose of the projected/designed system. The components of this vector are the continuous parameters of targets, which are subject to measurement, such as range or rate, and also the discrete/digital solutions about the situation as a whole - presence

of targets, their number, type, etc. Evaluation/estimate  $\hat{z}$ , obtained as a result of processing the echo signals, differs from true beam vector  $z$  and is random variable, which depends on the concrete/specific/actual realization of signals and interferences. RLS implements its designation/purpose the better, the nearer the evaluation/estimate  $\hat{z}$  to the true value of  $z$ , moreover here it goes without saying it is necessary to have in mind averaging on many realizations, and also, possibly, to give varying "weight" to the different components of vector  $z$ . Therefore as the criterion of the quality of system it is possible to select certain adequate/approaching functional, depending on difference  $\hat{z} - z$ . Not stopping in more detail on a question (generally speaking, important) about the appropriate structure of this functional, let us note that as the measure of quality it can serve, for example, the mean square of the difference

$$\epsilon = \overline{|\hat{z} - z|^2}, \quad (V)$$

where the feature designates averaging on many realizations of signals and interferences for the concrete/specific/actual situation, or, possibly, on many situations.

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The problem of synthesis consists in this selection of characteristics of RLS so that the value  $\epsilon$  would be minimum.

This approach to the synthesis is straight line in the sense that the criterion of quality (V) considers final effect and it is directly connected with equipment usage. However, there is a series/row of the reasons, due to which this approach in practice is not used.

In fact, we only formalized to some degree the problem of design of RLS as a whole. Minimized value  $\epsilon$  depends on all important characteristics of system - the form of the sounding signal, pulse reaction of receiver, antenna radiation pattern and algorithm of further processing. All these characteristics substantially affect the quality of the execution of tasks and they all, according to the previous setting, they must be optimized together, taking into account mutual effects. Of course of this consists strictly optimum design. But the problems of a similar scale not randomly are solved usually on the base of engineering intuition, and not analytical methods. To regularize the solution of this problem is completely impossible and even, in our opinion, it is not always expedient.

After stepping back from the strict approach indicated, they dismember task on the part, as far as possible selecting the locked groups of questions. Thus, the problem of angular measurements,

connected with the design of antenna, is examined independent of rangings and rate where the main role play waveform and method of reception/procedure<sup>1</sup>.

FOOTNOTE 1. With the super wide-band signals angular measurements and measurements of a range-speed are not independent variables and must be examined together [79]. ENDFOOTNOTE.

In the separate group are usually selected also the requirements, connected with the concrete/specific/actual designation/purpose of system and which affect in essence the algorithm of treatment, in our terms - to the structure of operator Z. Other at the same time requirements, which affect, mainly, to operator Y, to a considerable degree are general/common/total for all locating systems. The same, for example, is the requirement of the maximum probability of the target detection or corresponding resolution from the measured parameter.

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Similar "particular" requirements relate to the separate nodes, but not to the system as a whole, and on their basis are revealed/detected the "working" criteria of optimum character, utilized during the design of the counterparts of the equipment. Loc

us note that the "particular" criteria indicated in any way always cannot be connected quantitatively with the general/common/total criterion of a quality of the type (V). Applying such criteria, we are based only on the approximate estimate of that how it affects, let us say resolution to the accomplishment of the final objective of system. Furthermore, very concept of resolution and analogous characteristics allows/assumes different interpretations, which affect, generally speaking, the results of synthesis<sup>1</sup>.

FOOTNOTE 1. Let us mention the "classical" concept of resolution, known already to Rayleigh, but just as valuable today [83], or its contemporary definitions, which are based on the statistical treatment of the tasks of detection and measuring parameters [19].  
ENDFOOTNOTE.

All this shows that unavoidable virtually rejection from a strict, general/common/total criterion (V) and the transition to the particular criteria, which characterizes the quality of separate devices/equipment, is always connected with certain risk.

But we will also use the particular criteria of quality, which consider only the properties of signals  $y(t)$  at the output of receiver and disregarding the subsequent processing. In other words, we limit our analysis by operator  $Y$  and wish to indicate sufficiently

general/common/total requirements on it without taking into account the subsequent operator Z. Here also there are several approaches, most substantiated from which, apparently, is the following.

Being returned at the beginning of the previous section, we will consider that for certain situation the three-dimensional/space density of "useful" objects to eat  $a(r)$ , and the density of the mixing reflectors and sources of additive interferences -  $b(r)$  and  $n(r)$  respectively. As it was noted, these values were by chance, but it is assumed that their probabilistic distributions are known. The aforesaid indicates, for example, that in certain region of space supposedly are "useful" targets with the known middle reflecting surface.

Due to the different kind of fluctuations and interferences these targets either will be discovered by RLS or no. The probability of detection affect also characteristic RLS - waveform, the diagram of antenna and the pulse reaction of receiver, and is placed the task of this selection of these characteristics, so that the probability of detection would be maximum let us assume with the assigned probability of false alarms.

Somewhat less strict, but close one is the requirement of the greatest excess (on the average) of useful signal at the output of the receiver above the level of the mixing reflections and other interferences. In the designations of the previous section (see (IIIb)) for the criterion of quality of RLS in this case is accepted value

$$\rho = \overline{y_s^2 (y_b + y_n)^2},$$

which it is necessary to maximize on arguments  $s(t)$ ,  $g(r)$  and  $h(t)$ .

This approach to the synthesis is sufficiently productive. It is used, for example, in the works of Spafford and Stutt [69, 71], and also Yakovlev [89]. Upon this formulation of the problem "automatically" are considered and are optimized the resolving properties of signals, they in the best way are coordinated with the task of the isolation of useful reflections of all others for the selected situation, moreover even does not appear the needs for introducing and defining the concept of resolution (as it was noted, this can be done differently). We will use this approach for solving one of the tasks of the synthesis of signals in chapter 6.

The important result, obtained on the base of this approach, consists, in particular, of the fact that the matched filter is optimum receiver only in cases when additive interferences of the type of white noise prevail above the mixing reflections. But if the

level of the latter is relatively great, the structure of optimum signal and optimum receiver is more complicated and it depends substantially on the concrete/specific/actual situation.

Specifically, this special feature/peculiarity blocks the widespread introduction of this approach to the synthesis. The necessary a priori information about the concrete/specific/actual situation - predicted mutual location of the useful and mixing objects, a level and the character of interferences and so forth it is possible, apparently to obtain and to in proper time use only in adaptive RLS or analogous devices/equipment, which make it possible to operationally change fundamental characteristics with changes in the situation. In connection with such devices/equipment the method in question will be, it is necessary to assume/set, that prevail.

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But in the application/appendix to the usual, not adaptive systems of the advantage of this approach they can become its deficiencies/lacks, since for the situation, which was not being assumed with the synthesis, the obtained solutions can prove to be very distant from the optimum.

The aforesaid relates also to the straight/direct criterion of

synthesis (V), since here is required the knowledge (in the statistical sense) the observed situation. Therefore not only the practical need for disengaging the task of the design of system as a whole to the individual parts, but also the absence of reliable information about the situation blocks the use/application of this direct method<sup>1</sup>.

FOOTNOTE <sup>1</sup>. Let us note one additional approach to the synthesis of signals, which also uses further a priori information but this time about the special features/peculiarities of the motion of target [64]. ENDFOOTNOTE.

It is possible to note two bypass routes of this difficulty. First, relying on the play treatment of problem, it is possible to attempt to determine the worst situation during which the probability of target detection is minimum, and to optimize the system characteristics for this situation. Besides the obvious complexity of this task let us note certain of its artificiality: in any way in all uses/applications of FLS is justified the assumption about the sufficiently great possibilities of each "player". Furthermore, if we allow such possibilities, task, apparently, will be reduced to a certain trivial situation of the type of detection against the background of white noise.

The alternate path is based on the assumption that there are characteristic, typical, the situations which are encountered comparatively frequently, and they by hypothesis they are sufficient for the development/detection of the optimum characteristics of equipment. In fact, precisely, this assumption is the basis of those placed classical of the tasks about the detection of the signal of known form or about permission/resolution of two or more similar signals against the background of interferences.

These research has as a goal to optimize certain part of the equipment (usually receiver response) for the typical situation. In the case of detection this situation assumes the presence of one pinpoint target against the background, for example, of white noise, in the case of permission/resolution - several close-together targets. In this case is used a strict statistical criterion of the optimum character, when the measure of quality is the probability of the correct solutions about the situation - about the presence of targets (detection) or about their number (permission/resolution).

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But for the typical situations indicated this approach naturally is closed with other, indirect and known long before the development of the statistical methods of synthesis.

Actually/really, from the solution of the mentioned statistical problems it follows that there are two groups of values, which affect the probability of the correct solutions. The first group includes, for example, relation the signal/noise. These values are directly connected with the probabilistic nature of radar surveillance and their role is correctly revealed/detected only during the statistical analysis. The characteristic representative of the second group is apparatus function of RLS, i.e., its response to the single pinpoint target in the absence of interferences. Apparatus function this is deterministic of its nature characteristic whose concrete/specific/actual structure depends only on the type and the parameters of RLS. Moreover for the typical situations indicated the optimizable characteristics of equipment - waveform, the antenna radiation pattern and the pulse reaction of receiver - affect the probability of the correct solutions not directly, but through changes in the apparatus function. For this very reason apparatus function can with a sufficient foundation serve as the object of synthesis<sup>1</sup>.

FOOTNOTE <sup>1</sup>. The aforesaid is correct in the more general case. As showed Spafford [69], the excess of the signal above the interference for the arbitrary situation depends only on the function of the

uncertainty/indeterminacy (apparatus function - see below), but not from the waveform or frequency receiver response individually.

ENDFOOTNOTE.

The corresponding deterministic approach to the synthesis, which uses a concept of equipment function, in greater detail is considered below. Now, summarizing the aforesaid, it is possible to note that in proportion to unavoidable simplifications in the straight line and the general/common/total approach to the optimization of equipment is substituted by less general/common/total, based on the analysis of characteristic, typical situations. But this, in turn, frequently it leads to the fact that straight/direct probabilistic evaluation criteria of quality proves to be more or less equivalent to indirect deterministic criterion, know it is considerably earlier. The latter, although has lax, in a sense heuristic, character leads in many instances to the results which obtain only further confirmation with the help of the probabilistic methods.

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Synthesis according to the apparatus function.

The apparatus function of measuring meter this is its response to pulsed input effect, which has the character of delta-function.

Initially the concept of apparatus function related only to the optical instruments - telescope, the microscope and by other, and in the application/appendix to them she was used already by Rayleigh, but with the same foundation this concept is applicable to the linear instruments, which measure any physical quantities.

By linear is understood the measuring meter whose output response  $\eta$  is connected with the input effect  $\xi$  by linear integral transform. In one measured parameter  $t$  this conversion takes the form

$$\eta(t) = \int_{-\infty}^{\infty} \xi(t') \chi(t-t') dt'. \quad (VI)$$

Kernel  $\chi(t)$  is an apparatus function, since, as can easily be seen,  $\eta(t) = \chi(t)$  for the case of impulsive effect  $\xi(t) = \delta(t)$ . Let us dismantle/select some elucidating examples.

Of course conversion (VI) is implemented by linear electrical circuit, in particular by receiver of RLS. In this case is measured time  $t$  of the entrance of input signal  $\xi(t)$ , and  $\eta(t)$  is an input signal of receiver. Apparatus function  $\chi(t)$  is its pulse reaction. The wider, is more prolonged, pulse reaction, the more strongly is distorted the input signal and the rougher other conditions being equal occur the measurements of the time of arrival.

In the case of the optical instrument  $t$  is a

three-dimensional/space (angular) coordinate, and  $\xi(\tau)$  and  $\eta(t)$  - brightness distribution in the plane of object and in image plane. Apparatus function  $x(t)$  is the point-source image, obtained taking into account to the diffraction also of other distortions, i.e., again the response of instrument to the input effect of the type of the delta-function (now delta-function corresponds to signal at the fixed point of space, but not at the specific moment of time).

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The extent of apparatus function characterizes the width of elementary spot on the shield of instrument; the larger/coarser the spot, the greater the introduced by instrument distortions and the more roughly is measured the position of the light source.

Let us consider even frequency measurements, for example, with the help of the usual wavemeter. Let to the input of wavemeter be supplied the monochromatic signal, i.e., input effect has a character of delta-function on the axis of frequencies. Reconstructing wavemeter, is fixed/recorded the response of instrument to this effect which will be, obviously, the resonance characteristic of wavemeter. If we study the serrated signal, which contains many harmonic components, with the retuning of wavemeter is obtained the compound curve, which is by the imposition of elementary responses.

This curve is the spectrum of signal, measured with some distortions, the larger, the wider the resonance characteristic. It is not difficult to see that this curve also corresponds to integral (VI) if we by  $t$  understand frequency, by  $\chi(t)$  - the resonance characteristic of wavemeter (strictly speaking, dynamic), and by  $\xi(t)$  and  $\eta(t)$  - the true and measured spectra respectively.

In all examples examined the resolution of instrument the higher, the less the extent of apparatus function. The corresponding determination of resolution as the extents of apparatus function, was introduced by Rayleigh [83]. Although it has deterministic character and is not considered the statistical nature of measuring errors, this determination frequently is used, since the "fine structure" of input effect is distinguished the better, the less the extent of apparatus function.

But not always the task of measurement requires the undistorted transfer of input effect. For example, it is possible to be interested only in the fact such as energy of signal is included in the assigned frequency band. Then arises a question about the synthesis, the construction of the instrument whose frequency apparatus function (resonance characteristic) has rectangular form with the assigned extent. Of course using usual terminology, here one should speak about the synthesis of band-pass filter. Consequently,

the synthesis of filters according to the frequency characteristic can be treated as synthesis according to the apparatus function.

From somewhat smaller foundation, but similarly it is possible to treat the synthesis of the circuits, which form the signals of special form.

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In this case is used the pulse reaction of circuit, i.e., apparatus function in the temporary/time, but not in the frequency representation.

Passing to the radar, we will consider RLS as the linear instrument, which measures the angular coordinates, range (delay time) and the radial velocity (Doppler frequency) of the reflecting objects. During this treatment apparatus function of RLS is its response, reaction to the single pinpoint target which corresponds, obviously, to impulsive input effect. From relationships/ratios (Ia) and (II), the describing conversions signals in the linear devices/equipment of RLS, follows the concrete/specific/actual structure of apparatus function, which let us write out here without taking into account some unessential factors:

$$\chi(r, t, \Omega) = g^1(r) \int_{-\infty}^{\infty} s(t') h(t - t') e^{i\Omega t'} dt'. \quad (VIIa)$$

It is clear that if we are interested only in angular measurements, apparatus function of RLS there will be the antenna radiation pattern (according to the power)

$$\chi(r) = g^2(r). \quad (\text{VII 6})$$

But if we concentrate attention in rangings and rate, the significant role plays joint apparatus function of coordinates  $t$  and  $\Omega$ :

$$\chi(t, \Omega) = \int_{-\infty}^{\infty} s(t') h(t - t') e^{j\Omega t'} dt'. \quad (\text{VIIa})$$

It depends on the form of the sounding signal  $s(t)$  and the pulse reaction of receiver  $h(t)$  and is called in the theory of radar the cross function of uncertainty/indeterminacy (cross-ambiguity function).

Principal value takes the particular form of this function, when receiver of RLS is matched filter.

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In these cases pulse reaction is connected with the sounding signal with relationship/ratio:

$$h(t) = s^*(-t),$$

and we come to the function of Woodward's uncertainty/indeterminacy,

called also the eigenfunction of uncertainty/indeterminacy (auto-ambiguity function) and which depends only on the waveform:

$$\chi_s(t, \Omega) = \int_{-\infty}^{\infty} s(t') s^*(t' - t) e^{i\Omega t'} dt'. \quad (\text{VIIr})$$

FOOTNOTE 1. By asterisk are here and throughout designated compositely conjugate values. ENDFOOTNOTE.

Finally, for the limited applications of radar Doppler target speeds can be considered negligible. Then RLS is used only as range finder, and the function of uncertainty/indeterminacy is converted into the autocorrelation function of the form

$$R(t) = \chi_s(t, 0) = \int_{-\infty}^{\infty} s(t') s^*(t' - t) dt'. \quad (\text{VIIa})$$

The matched filtration is the optimum method of the reception of the echo signals in the sense that in this case is reached the greatest probability of detection against the background of white noise. For this very reason the function of the uncertainty/indeterminacy of Woodward and autocorrelation function, that assume this type of receiver, have so high a value in the theory of radar (in particular as the criteria of the quality of signals). But in light of the aforesaid earlier it is possible to emphasize that the use/application of apparatus functions in the tasks of synthesis always assumes particular situation - impulsive input

effect on the measuring meter. The use of function of Woodward's uncertainty/indeterminacy is assumed also that the observation occurs against the background of the additive white noise, which prevails above all other interferences [otherwise matched filter is not optimum receiver, and should be used the cross function of uncertainty/indeterminacy (VII c)].

However, we see that the synthesis of signals according to the functions of uncertainty/indeterminacy or according to the autocorrelation functions is justified completely in the same measure, in which is justified the synthesis of antennas according to the radiation patterns or the synthesis of filters according to the frequency characteristics. In all these cases are used the deterministic criteria of quality - apparatus functions - instead of stricter statistical criteria.

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In a strict setting as the criterion of synthesis must serve the probability of isolation or evaluation of the parameters of useful signals against the background of those mixing, when the signals indicated were distributed somehow in the appropriate interval of angles (synthesis of antennas), in the frequency domain (synthesis of filters), on the plane time - frequency (synthesis of signals for

RLS, which measure the range and the rate) or, finally, only in the time (synthesis of signals for the range finder).

In the principle statistical approach always offers further possibilities in comparison with the synthesis according to the apparatus functions. Thus, with the synthesis of antenna it is possible, for example, to attempt to fulfill the mutual compensation for interferences from different sources, which arrive on the different minor lobes of diagram. In some cases this is possible, although is required, obviously, the very complete knowledge of concrete/specific/actual situation. But usually we use the simpler and more universal method of synthesis, being given in a certain adequate/approaching manner very radiation pattern - apparatus function for the angular measurements. In this case are considered the actual conditions for the work of system in that measure, in which they are frequently known with a sufficient reliability. In view of such conditions we are given for some systems the highly directional, "pencil" diagram, for others - the diagram of special form, for example, cosecant. The same approach is used for the synthesis of the filters when we choose the "adequate/approaching" frequency characteristic - apparatus function for the frequency measurements, although, strictly speaking, seaboard would be solve the statistical problems about the isolation of the signals of different frequencies from the interferences.

The aforesaid entirely relates to the synthesis of signals. Being given the desired structure of the corresponding apparatus functions - function of uncertainty/indeterminacy or autocorrelation - we consider the actual conditions for work of RLS, without overloading at the same time the problem of synthesis. The role of stricter, statistical methods is reduced in this case to the proof, the evaluation/estimate of the admissibility of this deterministic approach, moreover in our view, statistic studies sufficiently convincingly confirm its legitimacy.

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From the aforesaid clear also that the deterministic treatment of synthesis, which assumes the preliminary selection of the desired apparatus function, always leaves certain scope for the engineering intuition. Choosing one or the other desired function, designer uses his experience of the solution of analogous problems. The richer this experience, the more complete the understanding of possibilities and limitations, inherent in the projected/designed equipment, the better to it to match different, usually contradictory requirements and it is possible to take into account the special features/peculiarities of the equipment realization of its project.

Apparently, by the expensive of tests/samples and errors, successes and disappointments it must pass anyone who seeks the solutions of similar problems. Are too complicated these tasks, so that always it would be possible to arrive at the foreseeable solution, using only straight/direct, completely serial modes and without resorting to heuristic ones. Synthesis according to the apparatus functions exists, in a sense, this heuristic method.

Basic concepts of the theory of signals.

Signal.

By the sounding signal it follows, strictly speaking, to call the real function of time  $u(t) = A(t) \cos[\omega_0 t + \Phi(t)]$ , the determining form of the emitted oscillation/vibration. Here  $A(t)$  and  $\Phi(t)$  - laws of amplitude and phase modulation respectively. It is possible to consider that the signal has the final duration  $T$ , but this value requires certain refinement.

Even when the emitted signal is conveniently depicted in the form of infinite sequence, the solution about the presence and the parameters of target - range, of rate, the angular coordinates - is

always accepted on certain final packet of the echo pulses or on one impulse/momentum/pulse. Therefore, without decreasing generality, it is possible to bound the signal in question by the final duration  $T$ , but this value depends on equipment usage and method of information processing. Of further interest are rangings and target speed. Signal  $u(t)$  it is possible in this case to examine during one or the maximum of several repetition periods. At least, duration  $T$  does not exceed the time of the coherence of signal.

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Analytical signal.

The separation of the single real signal  $u(t)$  to the envelope  $A(t)$  and the fluctuating factor  $\cos [\omega_0 t + \phi(t)]$  also requires refinement. These factors it is possible to select more or less arbitrarily, retaining their product, in connection with which appears the difficulty with a strict formulation of concepts of amplitude and phase modulation. These concepts prove to be ambiguous.

The most substantiated way of eliminating this ambiguity leads to the introduction of analytical signal. In this case real function  $u(t)$  is supplemented by the imaginary component  $v(t)$ , so that is formed the complex signal  $s(t) = u(t) + jv(t)$ . Component  $v(t)$  it depends on  $u(t)$  and it is connected with it with the conversion of Gilbert:

$$v(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(t')}{t - t'} dt'.$$

This selection of function  $v(t)$  has the weighty physical foundations (appendix 2). Furthermore, for each cosinusoidal

component in spectrum  $\tilde{u}(\omega)$  function  $u(t)$  is the same in the amplitude harmonic component in spectrum  $\tilde{v}(\omega)$  of function  $v(t)$ . Therefore spectrum  $\tilde{s}(\omega)$  of the composite signal  $s(t)$  is different from zero only with the positive ones  $\omega$ , and in this region of the spectrum  $\tilde{u}(\omega)$  and  $\tilde{s}(\omega)$  coincide in form and are characterized by only the scale factor:

$$\tilde{s}(\omega) = \begin{cases} 2\tilde{u}(\omega) & \text{при } \omega > 0, \\ 0 & \text{при } \omega < 0. \end{cases}$$

Key: (1). with.

Thus, the conversion of Gilbert leads to the composite signal whose spectrum has the same functional structure, as the spectrum of initial real oscillation.

After registering analytical signal in the form

$$s(t) = A(t)e^{j\varphi(t)}, \quad (\text{VIIIa})$$

it is seen, that now enveloping and phase they are determined by the only form:

$$A(t) = \sqrt{u^2(t) + v^2(t)}; \quad \varphi(t) = \arctg \frac{v(t)}{u(t)},$$

in this case real part retains the assigned form

$$u(t) = \text{Res}(t) = A(t) \cos \varphi(t).$$

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This removes ambiguity in the determination of amplitude and phase

factors.

Composite enveloping.

If real function  $u(t)$  has rapidly oscillating character, then phase  $\phi$  usually can be presented in the form of sum of two the addend

$$\varphi(t) = \omega_0 t + \Phi(t),$$

moreover second of them is changed relatively slowly ( $\Phi'(t) \ll \omega_0$ ) and it characterizes phase modulation of signal. Generally speaking, the determination of the carrier frequency  $\omega_0$  and with respect to linear component of phase  $\phi(t)$  also requires refinement. But for many questions of the theory of signals linear component of phase does not play the significant role. Therefore we can use with composite signal amplitude envelope

$$s(t) = A(t)e^{i\varphi(t)}, \quad (\text{VIIIb})$$

disregarding the absence rapid oscillation factor  $e^{i\omega_0 t}$  and without being interested in the value of linear inphase component  $\Phi(t)$ .

For composite envelope (VIIIb) we retain the same designation, as for signal (VIIIa). Moreover, for the brevity we speak signal  $s(t)$ , understanding by this composite envelope (VIIIb) without taking into account the carrier frequency.

Spectrum of signal.

Transition/junction to composite envelope corresponds to the transfer of the spectrum from the carrier to the zero frequency. The spectrum of composite envelope (usually we call its spectrum of signal) is placed both with the positive ones and at the negative frequencies, and, if the carrier frequency  $\omega_0$  is sufficiently great, it is possible to consider that the spectrum is spread to entire frequency domain  $-\infty < \omega < \infty$ .

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After determining spectrum  $\tilde{s}(\omega)$  by the relationship/ratio

$$\tilde{s}(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt = a(\omega) e^{-j\alpha(\omega)},$$

it is called  $a(\omega)$  by the amplitude spectrum, and  $\alpha(\omega)$  - by the phase spectrum of signal.

Function of uncertainty/indeterminacy.

As it was noted, the function of uncertainty/indeterminacy is an apparatus function RLS in the coordinates "time - frequency". It characterizes response RLS to the single pinpoint target whose range corresponds to the moment/torque of time  $t=0$ , and speed - to Doppler

frequency  $\Omega$ . Distinguish the cross function of the uncertainty/indeterminacy

$$\begin{aligned} \chi_{sh}(t, \Omega) = \\ = \frac{1}{\sqrt{E_s E_h}} \int_{-\infty}^{\infty} s\left(t' + \frac{t}{2}\right) h^*\left(t' - \frac{t}{2}\right) e^{j\Omega t'} dt' \quad (\text{IXa}) \end{aligned}$$

and eigenfunction uncertainties/indeterminacies (Woodward)

$$\chi_s(t, \Omega) = \frac{1}{E_s} \int_{-\infty}^{\infty} s\left(t' + \frac{t}{2}\right) s^*\left(t' - \frac{t}{2}\right) e^{j\Omega t'} dt'. \quad (\text{IXb})$$

These expressions insignificantly differ from (VIIIc) and (VIId), but here the functions of uncertainty/indeterminacy are calibrated, on the basis of the condition

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\chi_{sh}(t, \Omega)|^2 dtd\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\chi_s(t, \Omega)|^2 dtd\Omega = 1.$$

The standardizing factor  $E$  is energy of the corresponding signal <sup>1</sup>, in particular

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt.$$

FOOTNOTE 1. It is more precise,  $E$  is the doubled energy of real signal, since components  $u(t)$  and  $v(t)$  possess equal energy.

ENDFOOTNOTE.

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Examining the signals, calibrated on the energy, i.e., after placing

$E=1$ , it is possible not to write out this factor. As is known, the functions of uncertainty/indeterminacy allow/assume also the equivalent recording through the spectra of the signals:

$$\chi_{sh}(t, \Omega) = \frac{1}{2\pi E} \int_{-\infty}^{\infty} \tilde{s}\left(\omega - \frac{\Omega}{2}\right) \tilde{s}^*\left(\omega + \frac{\Omega}{2}\right) e^{j\omega t} d\omega; \quad (IXc)$$

$$\chi_s(t, \Omega) = \frac{1}{2\pi E} \int_{-\infty}^{\infty} \tilde{s}\left(\omega - \frac{\Omega}{2}\right) \tilde{s}^*\left(\omega + \frac{\Omega}{2}\right) e^{j\omega t} d\omega. \quad (IXd)$$

Autocorrelation function.

If RLS is intended for the measurements only of the range of targets, then, as a rule, are used such signals, that Doppler frequency switches become negligible. Under these conditions the vital importance has only one section of the function of uncertainty/indeterminacy  $\chi(t, 0) \equiv R(t)$ . This function is called the autocorrelation function of signal; for it we have two equivalent expressions, which ensue from (IXc) and (IXd) with  $\Omega=0$ :

$$R(t) = \frac{1}{E} \int_{-\infty}^{\infty} s\left(t' + \frac{t}{2}\right) s^*\left(t' - \frac{t}{2}\right) dt'. \quad (Xa)$$

$$R(t) = \frac{1}{2\pi E} \int_{-\infty}^{\infty} |\tilde{s}(\omega)|^2 e^{j\omega t} d\omega. \quad (Xb)$$

As in the case of functioning the uncertainty/indeterminacy, here it is possible to use the standardized/normalized signals, after assuming  $E=1$ . From (Xb) it is clear that the autocorrelation function is completely determined by the spectrum of the power of signal  $|\tilde{s}(\omega)|^2$  and in turn, determines this spectrum.

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Chapter 1.

#### CRITERION OF PROXIMITY.

In question in this chapter general/common/total approach is applicable to different tasks of the synthesis of signals, antennas, filters, etc. For these tasks it is characteristic that we attempt to find out the parameters of the synthesized object or, more generally, its structure, which ensures some desired properties. The class of permissible structure is always limited, since the objects of synthesis must permit realization under some specific conditions. Characteristic also that the desired properties are usually impracticable on the assigned class of structures. For example, they attempt to obtain, but do not obtain filters with the strictly table-shaped frequency characteristic or antennas without the minor lobes of radiation patterns.

In such cases, which are of fundamental interest, the synthesis of properties to finding of the optimum structure, which gives best approximation to the desired properties. It is assumed also that the desired properties (or property) are determined in a certain

deterministic manner, without resorting to the probabilistic description of task. In particular, as the desired property can be assigned the required antenna radiation pattern, the frequency characteristic of filter or the function of the uncertainty/indeterminancy of signal.

Let us attempt to give to a similar problem of synthesis the adequate/approaching mathematical description, and as the first space let us note its connection/communication with the task of approximation.

#### 1.1. Task of approximation.

In sufficiently general/common/total formulation this task consists in the following [1, 20, 24].

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Is given certain multitude of the  $X$  functions  $x(t)$ , and also function  $y(t)$ , which does not belong to set  $X$ . It is necessary to determine function  $x_{opt}(t) \in X$ , which provides best approximation to  $y(t)$ . The criteria of approximation/approach can be different. It is possible, for example, to require, so that would be minimum a quadratic difference in the functions in the assigned time interval

$(-T/2, +T/2)$ , i.e.

$$\int_{-T/2}^{T/2} |y(t) - x(t)|^2 dt = \min. \quad (1.1)$$

or minimize the great divergence of functions in the same interval

$$\max_{t \in T} |y(t) - x(t)| = \min. \quad (1.2)$$

The first condition corresponds to the quadratic criterion of approximation/approach, the second - to uniform (minimax) criterion. More general/more common/more total treatment is reduced to the following. Of each function  $x(t)$ , which belongs to set  $X$ , is placed in the conformity certain non-negative number  $d(x, y)$ , which depends also on assigned function  $y(t)$ . The condition for the best approximation/approach consists in the minimization of value  $d(x, y)$  on all elements of set  $X$ :

$$d_{\min} = \min_{x \in X} d(x, y). \quad (1.3)$$

Value  $d_{\min}$  depends on function  $y(t)$  and set  $X$  and characterizes the quality of the best approximation on this set. Different criteria of approximation/approach, in particular mentioned quadratic and minimax, are determined by the rule, according to which the pair of functions  $x(t)$ ,  $y(t)$  is compared with number  $d(x, y)$ . A change in this rule leads not only to different values  $d_{\min}$ , but in general and to different approximating functions  $x_{\text{opt}}(t)$ .

Value  $d(x, y)$  is called the distance between functions  $x(t)$  and  $y(t)$ , the mentioned rule, which is determining this value,

characterizes the metric of certain abstract space. These important concepts lead to the geometric interpretation of the task of approximation and many tasks of synthesis. In greater detail let us pause at the mathematical essence of the concepts indicated.

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## 1.2. Simplest concepts of functional analysis.

In many  $H$  elements/cells of arbitrary nature  $(x, y, z, \dots)$  the non-negative value  $d(x, y)$  is called distance, if it satisfies the following axioms of metric:

1)  $d(x, y) = 0$  when and only when  $x = y$  - axiom of identity;

2)  $d(x, y) = d(y, x)$  - the axiom of symmetry;

3)  $d(x, z) \leq d(x, y) + d(y, z)$  - the triangle axiom.

Set itself  $H$  is called the locked metric space, if to each pair of its elements/cells is set in the conformity distance  $d(x, y)$ , which satisfies the axioms indicated, and set  $H$  contains all elements/cells, for which specifically is distance, switching on all maximum elements/cells. Element/cell  $x$  is called the limit of

sequence  $x_1, x_2, \dots, x_n, \dots$ , or, it is shorter, by the maximum element/cell of space, if  $d(x, x_n) \rightarrow 0$  with  $n \rightarrow \infty$ .

Any set  $X$ , entering  $H$ , is subspace or region of space  $H$ .

The given determinations consider the most general/most common/most total properties of distance and space and they are the natural generalization of the properties of usual three-dimensional space. Stressing analogy with the geometric forms, the elements/cells of metric spaces frequently call points.

The dominant role in the functional analysis and its applications/appendices play such spaces, in which are additionally determined the operations of addition of elements/cells and their multiplication by real or complex numbers, moreover both operations satisfy the normal conditions of commutativity, associativity and distributivity. Such spaces are called linear.

If, furthermore for each element/cell  $x$  of linear space is determined norm  $\|x\|$ , which satisfies the following axioms:

- 1)  $\|x\| \geq 0$  and  $\|x\| = 0$  only if  $x = 0$ ;
- 2)  $\|x + y\| \leq \|x\| + \|y\|$ ;
- 3)  $\|\alpha x\| = |\alpha| \|x\|$ .

Key: (1). moreover. (2). only if.

the space is called standardized/normalized. It is obvious, norm is the generalization of the length of vector in the usual space.

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The linear standardized/normalized space becomes metric, if distance in the form

$$d(x, y) = \|x - y\|,$$

which also specifically corresponds to usual three-dimensional space.

Finally, if in the linear standardized/normalized metric space  $H$  is determined the scalar product of elements/cells  $(x, y)$  satisfying conditions:

1)  $(x, y) = (y, x)^*$ ;

2)  $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$ ;

3)  $(\lambda x, y) = \lambda(x, y)$ , where  $\lambda$  - any complex number, and norm  $\|x\|$  is connected with the scalar product with the relationship/ratio

$$\|x\|^2 = (x, x),$$

the space is called Hilbert.

The concept of a Hilbert space is sufficiently

general/common/total, it is added to appropriate multitudes of functions from one or several variable/alternating, to many vectors, matrices/dies, numerical or functional sequences, etc. At the same time whatever nature had the elements/cells in question, then it is possible to liken to the points of space, after preserving analogy with the geometric forms. In this case many dependences and properties of the objects in question usually obtain demonstrative geometric description, which simplifies the solution of series of problems. We will attempt not to use geometric analogies for the proof of fundamental results, but they us will help to plan the methods of solution, to explain the essential features of the tasks of synthesis. Specifically, of this consists the principal value of the concepts of functional analysis for this work.

### 1.3. Space of signals.

In certain cases of the concept of distance, norm, space and so forth it is possible to introduce completely naturally, but not axiomatic, as it is done above. Let us consider, for example, many signals of the limited energy, i.e., many functions  $s(t)$  with the integrated square

$$E = \int_{-\infty}^{+\infty} |s(t)|^2 dt < \infty.$$

After selecting certain orthonormal set of functions  $f_1(t), f_2(t), \dots, f_n(t), \dots$ , we can present signal as expansion

$$s(t) = \sum_{n=1}^{\infty} s_n f_n(t). \quad (1.4)$$

Then function  $s(t)$  is completely assigned by the set/dialing of numbers - the coefficients of expansion  $s = (s_1, s_2, \dots, s_n, \dots)$ . This ordered sequence of numbers can be treated as multidimensional vector, and numbers themselves  $s_n$  - as the projections of vector on some axes in the multidimensional space. After defining further distance, norm and scalar product by the relationships/ratios, similar to usual three-dimensional space, i.e., after placing

$$\begin{aligned} d(s_1, s_2) &= \left\{ \sum_n |s_{1n} - s_{2n}|^2 \right\}^{1/2}, \\ \|s\| &= \left\{ \sum_n |s_n|^2 \right\}^{1/2}, \\ (s_1, s_2) &= \sum_n s_{1n} s_{2n}^*. \end{aligned} \quad (1.5)$$

we satisfy (as it is not difficult to check) all axioms indicated above. Consequently, many multidimensional vectors  $s$  (or, which is the same thing, many ordered numerical sequences  $s_1, s_2, \dots, s_n, \dots$ ) are Hilbert space. In the functional analysis such space frequently designate  $l^2$ . This representation of signals - as vectors in the Hilbert space - is used extensively, for example, in the theory of freedom from interference for the geometric description of the corresponding tasks.

We introduced the values indicated, using the expansion of

functions  $s(t)$  in the Fourier series (1.4) and considering base set of functions  $f_n(t)$  as certain coordinate system, and coefficients  $s_n$  - as projections on the corresponding axes.

However, in the usual space all geometric concepts can be directly connected with the parameters of vectors, without resorting to coordinate representation.

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Similar to this, in the generalized space in question it is possible to express distance, norm and scalar product directly through the functions of time, without using their expansions in the Fourier series. For this it suffices to use to the previous relationships/ratios equality Parseval for series/row (1.4). As a result it is obtained

$$\begin{aligned} d(s_1, s_2) &= \left\{ \int_{-\infty}^{+\infty} |s_1(t) - s_2(t)|^2 dt \right\}^{1/2}; \\ \|s\| &= \left\{ \int_{-\infty}^{+\infty} |s(t)|^2 dt \right\}^{1/2}; \\ (s_1, s_2) &= \int_{-\infty}^{+\infty} s_1(t) s_2^*(t) dt. \end{aligned} \quad (1.6)$$

Consequently, these values do not depend on the selection of the system of base functions  $f_n(t)$ . Transition/junction from one system to another changes the coefficients of expansion  $s_n$ , but their

combinations, which express distance, norm and scalar product are invariant during such conversions and are determined only by the structure of signals. Here also there is an analogy with the usual three-dimensional space: the length of vector, the distance between the vectors and so forth they are expressed as projections on the axis, but they do not depend on the selection of coordinate system.

Values (1.6) satisfy the axioms of Hilbert space. This space of signals - the space of the quadratically summarized functions - frequently designate  $L^2$ .

With axiomatic formal approach of space  $l^2$  and  $L^2$  - these are different spaces. Elements/cells of one of them are numerical sequences, and another - function; distance, norm, scalar product they are expressed differently. But we obtained, obviously, only different descriptions, various forms of one and the same laws (similarly how Euclidean and analytical geometry they give only different description of one and the same mathematical essence).

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Here we come to the important concept of isometric spaces. If between the elements/cells of two spaces is established/installed one-to-one conformity, such, that the norms of equivalent components,

and also of distance and scalar products for the corresponding pairs of elements/cells are identical, then such spaces are called isometric. In view of equality of Parseval for the generalized series of Fourier (1.4) these conditions are satisfied in spaces  $L^2$  and  $L^2$ .

Isometric space are completely equivalent in the examination of the questions, which are the object/subject of this book; it is possible to use one or the other isometric space depending on convenience.

Fourier transform places in the conformity to each signal  $s(t)$  his spectrum  $\tilde{s}(\omega)$ :

$$\tilde{s}(\omega) = \int_{-\infty}^{+\infty} s(t) e^{-j\omega t} dt.$$

This conformity is mutually unambiguous, since also

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{s}(\omega) e^{j\omega t} d\omega.$$

Many spectra  $\tilde{s}(\omega)$  form Hilbert the space (which we further designate  $\tilde{H}$ ), if we determine distance, norm and scalar product by the relationships/ratios:

$$\begin{aligned} d(s_1, s_2) &= \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\tilde{s}_1(\omega) - \tilde{s}_2(\omega)|^2 d\omega \right\}^{1/2}; \\ \|s\| &= \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\tilde{s}(\omega)|^2 d\omega \right\}^{1/2}; \\ (s_1, s_2) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{s}_1(\omega) \tilde{s}_2^*(\omega) d\omega. \end{aligned} \quad (1.7)$$

Equality Parseval (for the Fourier integrals, but not for the series/rows of the type (1.4)) shows that the corresponding values in spaces  $H$  and  $\tilde{H}$  coincide, i.e., spaces are isometric. This it indicates the equivalence of the representations of signals in the form of the functions of time and in the form of the spectra - the functions of frequency.

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However, complete equivalence occurs only if is applied the quadratic space metrics, which corresponds to formulas (1.5)-(1.6). If, let us say, is used uniform (Chebyshev) metric, i.e., distance is measured by the maximum divergence of functions in certain interval -

$$d(s_1, s_2) = \max_{t \in T} |s_1(t) - s_2(t)|, \quad (1.8)$$

that isometric nature it is not observed and the space of the spectra it is not equivalent to the space of signals.

## 1.4. Fundamental task of synthesis.

In the terms of the functional analysis (or, that almost the same, in the geometric terms) task examined above of approximation is formulated as follows. In function space  $H$  with metric  $d(x, y)$  is a region  $X$  whose points  $x \in X$  form many approximating functions. In the same space there is a function  $y$ , which does not belong to region  $X$ . It is necessary to determine point  $x_{opt} \in X$ , least distant (in sense of space metrics) from the given point  $y$  (Fig. 1.1) <sup>1</sup>.

FOOTNOTE <sup>1</sup>. Using conditional geometric model, we represent the elements/cells of multidimensional spaces as the points of plane. On figure to set  $X$  corresponds one-dimensional curve. By this it is stressed that a number of measurements for region  $X$  is frequently less than for entire space. ENDFOOTNOTE.

The criterion of approximation/approach depends on metric. quadratic metric (1.6) leads to the least squares criterion (1.1), Chebyshev metric (1.8) - to minimax criterion (1.2). An error in the approximation is measured by the minimum distance

$$d_{min} = \min_{x \in X} d(x, y) = d(X, y)$$

from point  $y$  to region  $X$ .

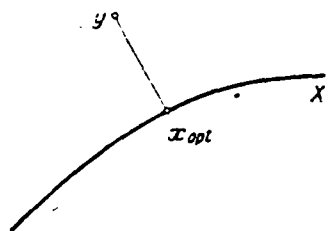


Fig. 1.1.

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The operator of finding the point of set  $X$ , nearest to  $y$ , is called the operator of design to this set and is designated  $P_X$ , so that

$$x_{opt} = P_X(y)$$

and

$$d_{min} = \min_{x \in X} d(x, y) = \|y - P_X(y)\|. \quad (1.9)$$

In certain cases of the problem of synthesis they are reduced and to a similar task. In the theory of electrical circuits is known, for example, the task about the forming two-terminal network when it is necessary to form the circuit whose impedance approximates the assigned function. Thus, for the impulse shaping, close to the rectangular ones, it is necessary to obtain approximation/approach to an impedance of the open section of long line, in our designations

$$y(\omega) = \operatorname{ctg} \frac{\omega \tau}{2},$$

where  $\tau$  - pulse duration. With the synthesis of circuits with the lumped parameters region  $X$  contains the functions of form  $x(\omega) = P_m(\omega)/Q_n(\omega)$ , where  $P_m$  and  $Q_n$  - polynomials, to which are superimposed also some further conditions.

Further frequently they resort to the artificial receptions/procedures. For example, it is possible to obtain approximation/approach to assigned  $y$ , if we use the expansion

$$\operatorname{ctg} z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - \pi^2 k^2},$$

and to be bounded to a finite number of terms of this series/row. Similar receptions/procedures lead also to other known results (see for example [26]). Thus is found out the rational-fractional function, which approximates with certain accuracy the assigned impedance, and on it is restored the electrical circuit of two-terminal network.

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Clear thus far questions about the criterion of approximation/approach and accuracy, let us note the following special feature/peculiarity of the task examined. Here to us it was completely known function - impedance of the segment of the long line approximation/approach to which was required to obtain. Specifically,

this made it possible to use approximation methods for the synthesis of circuit. However, considerably more frequent we do not have so perfect information about the desired structure of object.

For example, with the synthesis of filter frequently there is known only the required amplitude-frequency characteristic, i.e., the modulus/module of transmission factor; the phase response of filter can be arbitrary.

It is analogous, with the synthesis of antennas frequently is assigned only desired amplitude radiation pattern, but the phase structure of field does not play the significant role <sup>1</sup>.

FOOTNOTE <sup>1</sup>. In the series/row of cases it is necessary, on the contrary, to perform antenna with the assigned phase diagram with by arbitrary amplitude.

Similar tasks were called mixed problems of the synthesis of antennas [2]. ENDFOOTNOTE.

Said means that in the appropriate metric space is not an only element/cell  $y$ , which possesses the desired property, but certain set  $Y$ , in each element/cell of which inherently this property.

In particular, set  $Y$  can contain all filters with the assigned modulus/module of transmission factor or all antennas with the assigned amplitude radiation pattern <sup>2</sup>.

FOOTNOTE <sup>2</sup>. Here is disregarded the requirement of physical feasibility, so that the question can deal with hypothetical filters or hypothetical antennas. ENDFOOTNOTE.

We will see also, that many tasks of the synthesis of the signals of those characterizing by similar conditions. Therefore, applying for the concreteness the terminology of the theory of signals, let us formulate the following task, which generalizes task indicated above of the approximation:

In the space of signals  $H$  are given many  $X$  signals  $x(t)$ , which allow/assume realization in some specific conditions (many permissible signals), and also nonintersecting with a  $X$  multitude  $Y$  of signals  $y(t)$ , each of which possesses the assigned desired property (many desired signals). It is necessary to determine signal  $x_{opt}(t) \in X$ , which provides best approximation to the property, which is determining set  $Y$ .

This task let us name the fundamental task of synthesis. With an obvious change in the terminology those formulated conditions can relate to the synthesis of filters, antennas or units of another nature. Let us emphasize again that has in mind the approximation/approach to the property, general/common/total for all elements of set  $Y$ , but not to any concrete/specific/actual element/cell  $y \in Y$ .

Questions about accuracy and criterion of approximation/approach, let us again, clear clarify the general method of solving assigned mission. We will aid the simple heuristic consideration, based on the geometric treatment.

If we fix arbitrary signal  $y \in Y$ , then, after using approximation methods, it is possible to determine the shortest distance of  $d(X, y)$  between this signal and set  $X$ , and to also find permissible signal  $x = P_X(y) \in X$ , ensuring best approximation to selected  $y$  (Fig. 1.2). This approximation gives certain approximation/approach to any property of signal  $y$ , including to the desired property, general/common/total for all  $y \in Y$ .

However, if we vary signal  $y$ , being moved on the region (by curved)  $Y$ , and to monitor distance of nearest  $x \in X$ , then it is possible to come to light/detect/expose signal  $y_{opt}$ , arranged/located

on the shortest distance from region X. This signal makes it possible to obtain better approximation/approach on set of X in comparison with all other signals set Y - indeed precisely distance  $d(x, y)$  it is the measure of the quality of approximation/approach.

Since set Y contains all signals, which possess necessary property, and permissible is any signal of set X, logical to assume that precisely signal  $y_{opt}$  should be selected as the "sample/specimen" with the approximation. But the nearest to  $y_{opt}$  signal of set X is signal  $x_{opt}$  arranged/located on the shortest distance from set Y.

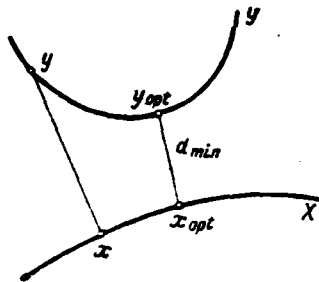


Fig. 1.2.

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As a result we come to the following position which subsequently is named the hypothesis (or criterion) of the proximity:

Solution of the fundamental problem of synthesis gives signal  $x_{opt} \in X$ , arranged/located on the shortest distance

$$d_{min} = \min_{\substack{x \in X \\ y \in Y}} d(x, y) \quad (1.10)$$

from set Y. Using operators of design on X and Y respectively, it is possible to register also

$$d_{min} = \min_{y \in Y} \|y - P_X(y)\| = \min_{x \in X} \|x - P_Y(x)\|. \quad (1.10a)$$

The formulated task of synthesis and the hypothesis of proximity are the basis of this work. Many questions of the synthesis of signals are reduced to this task or its generalizations, moreover the

hypothesis of proximity indicates the general/common/total way of experiment.

For the first time the hypothesis of proximity was formulated by one of the authors of this book in 1967 [8]. From the works of predecessors it is possible to note the following.

Landau and Pollack [43], examining the task, investigated by us in chapter 2, mention about the possible treatment of synthesis as to the problem of the minimization of the angle between the appropriate subspaces. This is close to our interpretation (see §1.7). Unfortunately, the more complete work of the same authors on the theme indicated (reference of 6 articles [43]) was not published.

In the number of research on pattern recognition (see for example [59]) as one of the heuristic algorithms of discrimination is mentioned that the called rule of proximity, which consists of the following. The tested object relates to that class, of which is less the distance (in the sense of certain space metrics of signs/criteria). Here it is possible to perceive analogy with our approach, but to another task, which differs significantly from the synthesis of signals, antennas or filters.

Let us note also that the theory of the synthesis of radar

signals is developed/processed comparatively recently, approximately/exemplarily from second half of the 1950th years. The synthesis of antennas and electrical circuits has, at least, thirty-year history. Under these conditions the appearance of a similar general/common/total idea in the theory of signals can be explained, perhaps, only by the fact that the latter is the branch of the theory of the freedom from interference where the geometric representations, analogous by that used by us, are used extensively.

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In connection with antennas or filters similar representations did not win acceptance that it could be reflected in the methods of synthesis.

#### 1.5. Some generalizations of fundamental task.

Above task of synthesis was formulated in the space of signals  $H$  or, it is more general/more common/more total, in the space, elements/cells of which are the objects of the synthesis of another nature.

However, a similar task can be formulated also in some other spaces, in connection with the elements/cells which are connected in

any manner with the objects of synthesis, but they are not identical to them. For example, with the synthesis of filter it is possible to examine as the objects not of the structure of quadrupoles, but their matrices/dies, transmission factors or let us assume transient functions. Similar versions are contained by the following diagram.

Let operator  $M$  place in accordance to each element/cell  $s$  of space  $H$  certain of his form  $s'$  in space  $H'$ :

$$s' = M(s); s \in H; s' \in H'.$$

Regions  $X$  and  $Y$  of space  $H$  are converted in this case into the new regions  $X'$  and  $Y'$  in space  $H'$ . It is obvious, with the synthesis it is possible to use a hypothesis of proximity in any of these spaces, and depending on what space is examined, the solution will be either element/cell  $x_{opt}$  of space  $H$  or element/cell  $x'_{opt}$  of space  $H'$  (Fig. 1.3).

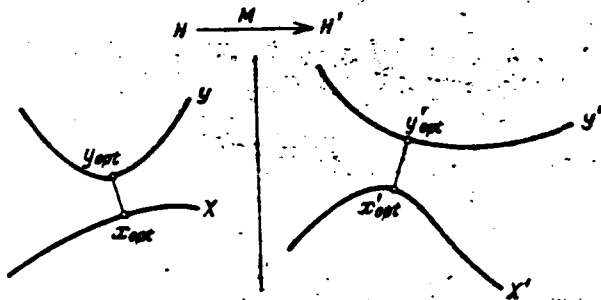


Fig. 1.3.

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These solutions are not equivalent, different conversions lead, generally speaking, to different tasks of synthesis and different solutions. Here it is expedient to consider three fundamental cases.

First case. Let spaces  $H$  and  $H'$  be isometric. As it was noted, this means that there is one-to-one conformity between the elements/cells and their forms, i.e., there is an inverse operator  $M^{-1}$ , which unambiguously reflects  $H'$  on  $H: s = M^{-1}(s')$ .

Furthermore, isometric conversion retains the distance between the appropriate pairs of the elements/cells:

$$\|s_1 - s_2\| = \|s'_1 - s'_2\|.$$

During this conversion sets  $X$  and  $Y$  do not change mutual

location. Therefore element/cell  $x'_{opt}$ , the realizing minimum of distance of  $Y'$  in space  $H'$ , is the form of element/cell  $x_{opt}$  of space  $H$ . Consequently, isometric conversions they do not lead to the new solutions of the problems of synthesis, all isometric spaces are equivalent in these tasks.

The second case occurs, if operator  $M$  realizes the homeomorphic conversion  $H$  on  $H'$ . This means that there is one-to-one conformity between  $s$  and  $s'$  (there is an inverse operator  $M^{-1}$ ), but the distances between the corresponding pairs of elements/cells are not equal to  $\|s_1 - s_2\| \neq \|s'_1 - s'_2\|$ . This conversion is equivalent to the elastic deformation of space. Actually/really, it is possible to introduce in the initial space  $H$  new metric, after assuming

$$d(s_1, s_2) = \|s'_1 - s'_2\|.$$

In view one-to-one conformity  $s$  and  $s'$ , and also that the homeomorphic conversion is mapping of a space  $H$  onto itself, new metric satisfies the necessary axioms. Therefore as a result of conversion some points converge, others, on the contrary, are separated/expanded, but does not occur mergings/coalescences of several points into one or discontinuous changes.

It is obvious, those points of regions  $X$  and  $Y$  which were located at the shortest distance from each other, after deformation they can not satisfy this condition. They will be replaced other

points, i.e., the solution of the problem of synthesis in space  $H'$  will be obtained other, than in space  $H$ .

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This change in the space metrics leads to a change in the criterion of approximation/approach, moreover for each criterion there is an adequate metric for which minimization of distance gives approximation/approach in the sense of the assigned criterion (see §1.6). By other words, there is a reversible operator  $M$ , which permits to convert the initial space  $H$  into the homeomorphic for it space  $H'$ , where use/application of a hypothesis of proximity gives the solution, matched with the assigned criterion of approximation/approach. However, as it will be clear, finding this operator it presents considerable difficulty.

High value for future reference has the third case of the conversions, during which space  $H$  is mapped not to entire space  $H'$ , but to certain part of it  $Q$ . Set  $X'$ , which reflects the set of the permissible objects, is included in this case within  $Q$  (Fig. 1.4).

Let us consider, for example, the transformation of signal into its autocorrelation function

$$s' = M(s) = R(t) = \frac{1}{E} \int_{-\infty}^{\infty} s\left(t' + \frac{t}{2}\right) s^*\left(t' - \frac{t}{2}\right) dt'. \quad (1.11)$$

Fourier transform from autocorrelation function is the energy spectrum of signal, i.e.

$$\tilde{R}(\omega) = |s(\omega)|^2.$$

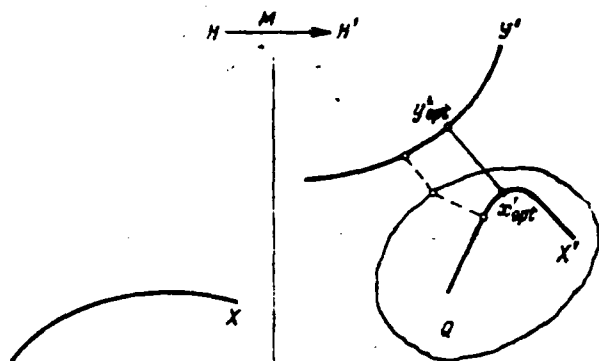


Fig. 1.4.

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Consequently, the spectrum of autocorrelation function is positive. This condition limits the class of the functions, feasible as autocorrelation, and are determined region  $Q$  of space  $H'$ , containing entire autocorrelation of function. But set  $Q$  does not cover/coat entire space  $H'$ . the elements/cells of this space are also the points, which are formed, for example, with the linear superposition of different autocorrelation functions. As a result are formed the functions with the arbitrary spectral density (not only positive), which supplement  $Q$  region to space  $H'$ .

The formulation of the fundamental task of synthesis assumes that the assigned property is feasible (since in space  $H$  is a set  $Y$ ,

in each element/cell of which inherently this property). However, in a number of cases it is expedient to be given as that required the impracticable property which possesses not one element/cell  $s \in H$ , and to seek best approximation to this property on the assigned set  $X$ . Specifically, in these cases are useful nonhomeomorphous transformations of the type in question.

Assume it is necessary to find out signal  $x \in X$  with a "good" autocorrelation function. It is thought that this function must be maximally crowded in the low time interval  $(-T/2, T/2)$  and have low remainders/residues out of this interval. In the absence of the more complete information about the necessary autocorrelation function  $R(t)$  it is expedient to take, for example, the condition

$$R(t) = \begin{cases} 1 & \text{при } -T/2 < t < T/2; \\ 0 & \text{при } |t| > T/2. \end{cases} \quad (1.12)$$

Key: (1). with.

But this autocorrelation function is impracticable <sup>1</sup>.

FOOTNOTE <sup>1</sup>. Since spectrum  $R(\omega)$  alternating. ENDFOOTNOTE.

It is not possible to indicate one signal  $y \in H$ , which possesses the property. However, after using transformation (1.11), it is possible to reformulate the task of synthesis in space  $H'$  - the space

of the autocorrelation functions where among the elements/cells, which supplement  $Q$  region, are those satisfying condition (1.12). Many such elements/cells of space  $H'$  let us designate, as earlier,  $Y'$ , but now it does not have a prototype in the space of signals  $H$ .

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As it is clear from Fig. 1.4, task is reduced to the minimization of the distance between sets  $X'$  and  $Y'$  space  $H'$ . Optimum is element/cell  $x'_{opt}$ , for which it is necessary to further find out prototype in region  $X \subset H$ .

Set  $X'$  can, in particular, cover/coast entire realizable region  $Q$ . Then we come to the task about the best approximation/approach to the assigned impracticable property on entire space of signals, representation/transformation of which is  $Q$  region. This is one of the most important of the problems of synthesis.

With the approximation/approach to the unrealizable property is applied also the following indirect method. First is found out optimum signal  $s_{opt}$ , ensuring best approximation to the assigned property in entire space of signals  $H$ , and then is realized approximation/approach to this optimum signal on the permissible set  $X$ . Since  $X$  is part of  $H$ , it is at first glance, this method is

correct: is solved the problem of synthesis in a broader class, and then is obtained best approximation on the assigned subset. However, this solution is reduced to two consecutive operations in space  $H'$ : first is determined the shortest distance between  $Y'$  and of  $Q$ , and then the corresponding point  $Q$  region is projected/designed for  $X'$  (dotted line in Fig. 1.4). As can be seen from figure, in general we do not come into that point  $X'_{opt}$ , which gives direct approximation/approach between  $Y'$  and  $X'$ . Therefore the indirect method indicated requires further proof.

From a practical point of view this method frequently can be justified by the fact that the desired property is not known completely accurately. In particular, condition (1.12) is formulated only on the base of intuitive considerations. It is possible to replace the required condition with certain close one to it and to use this possibility for simplification in the task. The replacement of impracticable condition by close one, but feasible makes it possible to formulate task in the initial space  $H$  (i.e. to arrive at the fundamental task of synthesis), which gives the considerable of simplification. The indirect method examined can be treated as one of the realizations of this possibility. Frequently it also happens, that the distance from  $Y'$  to  $Q$  is considerably more (or, on the contrary, it is considerably less) than from the appropriate (nearest to  $Y'$ ) point  $Q$  region to  $X'$ .

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For example, if region  $X'$  corresponds to set  $ChM$  of signals, -  $Q$  region to many arbitrary signals, and  $Y'$  is assigned by the desired (impracticable) autocorrelation function, then as it is possible to show, the distance between  $Y'$  and of  $Q$  remains final, and the distance between  $Q$  and  $X'$  asymptotically approaches zero during the large compression. Therefore with the synthesis of signals with the sufficiently large compression completely it is possible to use the indirect method indicated, moreover main role plays the first stage - the approximation/approach of the assigned autocorrelation function on many arbitrary signals (see Chapter 4).

#### 1.6. Criterion or hypothesis?

Upon the correct formulation of the problem of approximation it is necessary to assign not only desired function  $y(t)$  and many  $X$  approximating functions, but also the criterion of approximation/approach. In other words, it is necessary to clarify, in what sense unknown of function  $x_{opt}(t)$  must approach assigned  $y(t)$ . The criterion of approximation/approach is determined by a condition of the type

$$e(x, y) = \min, \quad (1.13)$$

where  $\epsilon$  - positive functional, and minimization is produced on all  $x \in X$ . Special cases (1.13) are conditions (1.1) and (1.2).

The criterion of approximation/approach, if it is assigned, usually makes it possible to establish/install the metric (it is more precise, quasi-metric) of space in which must be solved the task of approximation. For example, it is possible to assume

$$d(x, y) = \phi[\epsilon(x, y)]. \quad (1.14)$$

where  $\phi$  - arbitrary increasing function. In particular, metric (1.5) is connected with quadratic criterion (1.1) with relationship/ratio  $d = \epsilon^2$ , and Chebyshev metric (1.8) is connected with minimax criterion (1.2) with simplest dependence  $d = \epsilon$ .

The selection of the criterion of approximation/approach is almost always a difficult and disputable/debatable question. The cases when it is possible with the proper foundation to indicate, what kind approximation/approach is necessary, they are, it is faster, by exceptions/eliminations from the general rule. We already mentioned the task about the forming two-terminal network, which is reduced to the approximation of the impedance of the segment of long line.

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It is obvious that it is necessary to approach this ideal, but it is in no way clear, what approximations/approaches - minimax, quadratic or others - will give the best shape of pulse. Besides the fact that does mean the best? how to measure the divergences from the desired rectangular form?

Frequently only the intuitive considerations are used during this selection, or preference is given up to that criterion which more easily leads to the solution.

In the locating tasks the criterion of approximation/approach it is possible, in the principle, to establish relying on the statistical analysis of problem as a whole, the detection problems or measurement under conditions of cre or the other interferences. Some research of this type is [15]. But here, as when selecting of general/common/total approach to the synthesis, fundamental obstruction is connected with the incompleteness of the a priori information about the concrete/specific/actual situation <sup>1</sup>.

FOOTNOTE <sup>1</sup>. Should be distinguished the criteria of the approximations/approaches under discussion, and criteria quality (synthesis, optimization), that were being mentioned in input

chapter. The criteria of quality formulate main circuit of the task - to obtain approximation/approach to the assigned apparatus function, to ensure the maximum of the probability of detection, etc. The criteria of approximation/approach play more modest role. They make more precise some special features/peculiarities of the decided task, they indicate, what kind approximation/approach is required to obtain. ENDFOOTNOTE.

It is clear, for example, that for eliminating the masking action of close ones in the range of targets it is necessary to reduce the remainders/residues of autocorrelation function. But what criterion of the level of remainders/residues to take, does approach the decrease of the greatest remainder/residue (minimax criterion) or the average (quadratic criterion)? This depends on situation. If the mixing targets are comparatively rare, the greatest remainder/residue characterizes the worst case when useful signal interferes with one of that mixing. But if the mixing reflections are arranged/located sufficiently tightly (dipole cloud, the background of terrain echoes or sea), in each quantum of range occurs the imposition of many random signals, and is here appropriate quadratic criterion [15].

Thus, even in the tasks of approximation the selection of the criterion of approximation/approach must be produced on the base of those initial prerequisites/premises which led to the setting of

entire problem. Even more is complicated this question in the tasks of the synthesis when the object of approximation/approach is not accurately known, but is assigned only certain property, inherent in many objects, and approximation/approach to this property is required to obtain.

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Let  $H$  - be Hilbert space, elements/cells of which are the objects of synthesis, in particular, the space of signals. Set  $X$  includes all permissible objects, and  $Y$  - objects, which possess the desired property. We are interested in the specific property of objects, for example, by the autocorrelation function of signal. This means that there is an operator  $M$ , whom places in the conformity to each element/cell of space the property indicated. In particular, with the synthesis according to the autocorrelation function

$$M(s) = R(t) = \frac{1}{E} \int_{-\infty}^{\infty} s\left(t' + \frac{t}{2}\right) s^*\left(t' - \frac{t}{2}\right) dt'.$$

Operator  $M$  maps space  $H$  into another space  $H'$ . But, in contrast to the cases, examined earlier, set  $Y$  is converted in this case into one point space  $H'$ , since all  $y \in Y$  possess one and the same desired property

$$M(y \in Y) = M_0.$$

As a result of this conversion the synthesis is reduced to the approximation: in space  $H'$  it is necessary to find point  $x'_{opt}$  of set  $X'$ , nearest to point  $M_0$  (Fig. 1.5).

It is here assumed that the space metrics is matched with the assigned criterion of the approximation/approach

$$\varepsilon(x', M_0) = \min. \quad (1.15)$$

Of course the selection of criterion (1.15) is so/such difficult as with the usual approximation.

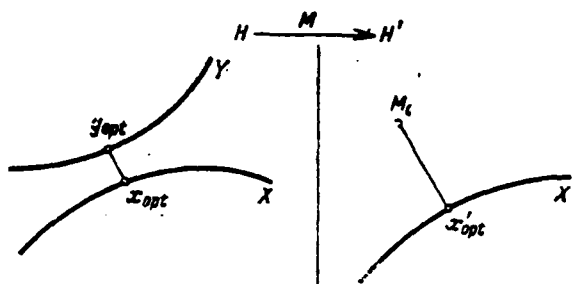


Fig. 1.5.

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But, furthermore, it is necessary on obtained form  $x'_{opt}$  to determine the unknown object, i.e., to return to space  $H$ .

With this, completely correct formulation, the problem of synthesis frequently proves to be extremely complicated. The hypothesis of proximity gives the simplified approach to the solution, which does not require mapping of a space  $H$ , but precisely this fact leads to certain contradiction in a question about the criterion of approximation/approach.

It is not difficult to establish/install, what condition satisfies object  $x_{opt}$  that obtained on the base of the hypothesis of proximity. As it is clear from Fig. 1.5 and formula (1.10), this

object realizes shortest distance in space  $H$  of set  $Y$ , i.e.,

$$d(x, Y) = \min_{y \in Y} d(x, y) = |x - P_Y(x)| = \min. \quad (1.16)$$

where the minimization is produced on all elements/cells  $x \in X$ .

Actually, this condition formulates the criterion of approximation/approach, utilized with our approach to the synthesis, and it is here appropriate to speak not about the hypothesis of proximity, but, rather, about the criterion of proximity. If space metrics  $H$  is fixed/recorded and in it there are many desired objects  $Y$ , then condition (1.16) it completely determines, what kind approximation/approach is achieved at the synthesis. In this sense our approach to the synthesis is reduced only to the special selection of the criterion of approximation/approach.

As it was noted, a change in the metric by the corresponding homeomorphic conversion is equivalent to the elastic deformation of space. This makes it possible to show that for each assigned criterion of approximation/approach there is an adequate metric with which the synthesis on the criterion of proximity leads to the same result, as direct synthesis on the assigned criterion.

Actually/really, let initial criterion (1.15) satisfy element/cell  $x'_{opt}$  of space  $H'$  (see Fig. 1.5). Being returned with the help of the inverse operator  $M^{-1}$  (ambiguous) into space  $H$ , let us establish that solution of problem gives element/cell  $x''_{opt}$  form of

which in space  $H'$  is  $x'_{opt}$ . In general  $x''_{opt}$  is not located at the shortest distance from  $Y$  in the initial space metrics  $H$ .

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But, after fulfilling the appropriate homeomorphic conversion, it is possible it goes without saying so to deform this space so that the element/cell  $x''_{opt}$  would prove to be nearest to  $Y$ . After such strain (it is obvious, not only) the use/application of a criterion of proximity (1.16) will be equivalent to synthesis on initial criterion (1.15).

Consequently, examining our approach to the synthesis on many different homeomorphic spaces, it is possible to speak about its universality in the sense that the criterion of proximity generalizes all other criteria of approximation/approach. Whatever initial criterion was assigned, there is always an adequate metric, which makes it possible to find the necessary solution on the base of the criterion of proximity.

However, the regular methods of finding this metric are unknown, and usually we forced to enter otherwise. We choose metrics of space  $H$  a priori, it is intuitive, and only in the course of solution of problem appears the possibility to check (via the analysis of

condition (1.16)), what criterion of approximation/approach satisfies the synthesized object. Of this consists the heuristic nature of the approach in question to the synthesis, this is why we speak not only about the criterion of proximity, but also about the hypothesis of proximity.

We will apply the hypothesis of proximity, mainly in the space with quadratic metric (1.6). This gives simplification in the solutions. But, in spite of the so/such "unjustified" selection of metric, condition (1.16) is reduced for the majority of problems to one of the commonly used ones or, at least, the acceptable criteria of approximation/approach. This means that in many instances we succeeds in confirming the applicability of the hypothesis of proximity in the simplest version, demonstrating the practical acceptability of the corresponding results.

In other cases when with the quadratic metric criterion (1.16) to justify is difficult, we attempt to indicate the adequate space metrics with which the criterion of proximity is equivalent to the given one. Frequently this can be done, but this path does not always lead to the practical results: in the new, "adequate" metric mathematical difficulties sharply they grow to find optimum signal, applying the criterion of proximity, it does not succeed. In such problems the solution, obtained according to the hypothesis of

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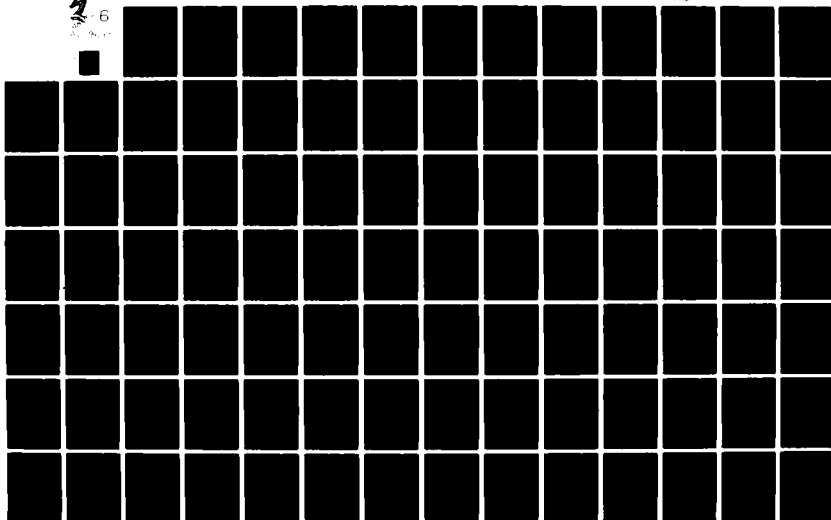
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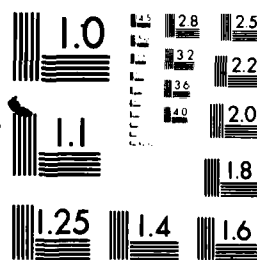
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proximity with the quadratic metric, is expedient to consider as initial approximation/approach and to obtain its refinement by one of the known iterative methods.

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Moreover, applying the latter, nothing interferes to use the initial criterion of approaching type (1.15). Specifically, this method of synthesis, which combines the analytical solution in the initial stage (based on the hypothesis of proximity) with the numerical iterations, is most efficient for the complex problems of synthesis.

#### 1.7. Standardization and the coefficient of proximity.

A question about the space metrics is connected also with the method of the standardization of signals. Usually the significant role plays form, but not signal amplitude, not its scale. All signals, which are characterized by only scale, i.e.,

$$s(t) = \mu f(t),$$

where  $f$  - assigned function,  $\mu$  - arbitrary positive value, frequently can be treated as one object of synthesis, since the scale does not affect those characteristics of signal which are important for permission/resolution or measuring the parameters of targets.

In this connection it is expedient to superimpose normalization the condition, which is unambiguously determining factor  $\mu$  according to function  $f(t)$ . This condition usually fixes/records energy of signal or its maximum amplitude. They assume/set

$$E = \int_{-\infty}^{\infty} |s(t)|^2 dt = 1 \quad (1.17)$$

or

$$s_{\max} = \max |s(t)| = 1. \quad (1.18)$$

In the linear standardized/normalized spaces two signals, that are characterized by only scale, are represented as the vectors of different length, directed along one straight line. On waveform depend the angular position of vector, while from the coefficient  $\mu$  - only its length, norm  $\|s\|$ . Condition (1.17) or (1.18) can be understood therefore as setting of the norm of signals. In other words, the signals, which satisfy standardization condition, are conveniently mapped by the points of the single hypersphere  $S$  in space  $H$ . Sets  $X$  and  $Y$  are in this case some sections of the surface of sphere. It is more precise,  $X$   $Y$  correspond to the conical spaces, shown in Fig. 1.6, but standardization condition satisfy only the traces of these cones on the surface of sphere  $S$ .

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Under these assumptions condition (1.17) means that the norm of signal must be determined by the relationship/ratio

$$\|s\| = \left\{ \int_{-\infty}^{\infty} |s(t)|^2 dt \right\}^{1/2};$$

analogously, condition (1.18) leads to the relationship/ratio

$$\|s\| = \max_t |s(t)|.$$

Norm determines, in turn, space metrics. Therefore standardization on energy (1.17) is conveniently used in space  $L^2$  with quadratic metric (1.6), and standardization in amplitude (1.13) - in the space with Chebyshev metric (1.8).

We examine the synthesis of radar signals according to the functions of uncertainty/indeterminacy or the autocorrelation functions. Both these functions use the signals, calibrated on the energy. Therefore the problems of synthesis it is frequently expedient to examine in space  $L^2$ , where condition (1.17) corresponds to standardization  $\|s\|=1$ .

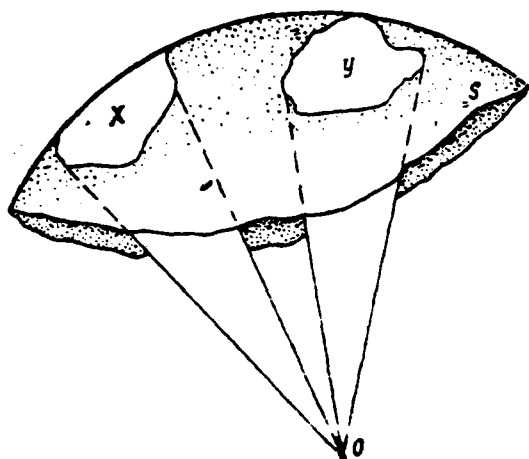


Fig. 1.6.

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Consequently, for the majority of the examined/considered by us problems is more convenient the quadratic, but not minimax criterion of approximation/approach, since precisely quadratic approximations are achieved by the minimization of distance to  $L^2$ .

Let two signals -  $x(t)$  and  $y(t)$  - have single norm, i.e., they satisfy condition (1.17). For the distance between them we have according to (1.6)

$$\begin{aligned}
 d^2(x, y) &= \int_{-\infty}^{\infty} |x(t) - y(t)|^2 dt = \int_{-\infty}^{\infty} [|x(t)|^2 + |y(t)|^2 - \\
 &\quad - 2\operatorname{Re} x(t) y^*(t)] dt = \|x\|^2 + \|y\|^2 - 2\operatorname{Re}(x, y) = \\
 &\quad = 2[1 - \operatorname{Re}(x, y)]. \quad (1.19)
 \end{aligned}$$

where  $(x, y)$  - scalar product.

It is evident that the minimization of distance of  $d(x, y)$  is equivalent to the maximization of the real part of scalar product  $(x, y)$ . This is not difficult to interpret geometrically. Angle  $\theta$  between vectors  $x$  and  $y$  is determined in the composite Hilbert space by relationship/ratio [43]

$$\cos \theta = \frac{\operatorname{Re}(x, y)}{\|x\| \cdot \|y\|}.$$

For standardized/normalized signals  $\|x\| = \|y\| = 1$ , therefore

$$\cos \theta = \operatorname{Re}(x, y).$$

It is obvious, the decrease of the distance between the unit vectors is equivalent to a decrease of the angle between them and to increase  $\cos \theta$ , that also corresponds to formula (1.19).

Value  $\cos \theta$ , which characterizes the distance between the standardized/normalized signals, plays in future large role. We will introduce for it the special designation  $C(x, y)$  and we will call the coefficient of the proximity of signals  $x$  and  $y$ . Thus,

$$C(x, y) = \operatorname{Re}(x, y) = \operatorname{Re} \int_{-\infty}^{\infty} x(t) y^*(t) dt. \quad (1.20)$$

moreover

$$\|x\| = \|y\| = 1.$$

The coefficient of proximity it is not difficult to express also through the spectra of signals.

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Actually/really, taking into account (1.7), we have

$$C(x, y) = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) \tilde{y}^*(\omega) d\omega. \quad (1.21)$$

Distance  $d(x, y)$  between the standardized/normalized signals is expressed as the coefficient of proximity by formula (1.19), i.e.,

$$d^2(x, y) = 2[1 - C(x, y)]. \quad (1.22)$$

Analogously is introduced the coefficient of proximity for a signal and certain multitude of signals, for example, for signal  $y$  and set  $X$ . This coefficient corresponds to distance (smallest) between  $y$  and  $X$ :

$$C(X, y) = \max_{x \in X} C(x, y). \quad (1.23)$$

Finally, the minimum distance between two sets  $X$  and  $Y$  also can be characterized by the coefficient of the proximity

$$C(X, Y) = \max_{\substack{x \in X \\ y \in Y}} C(x, y). \quad (1.24)$$

In the latter/last formulas it is assumed that sets  $X$  and  $Y$  are arranged/located on the surface of single hypersphere, i.e.,  $X, Y \subset S$ .

Using the introduced concepts, it is possible to formulate the hypothesis of proximity also as follows: solution of the fundamental

problem of synthesis gives signal  $x_{opt} \in X$ , the realizing coefficient of proximity  $C(X, Y)$  between regions  $X$  and  $Y$ .

In conclusion let us note that the coefficient of proximity is analogous to the correlation coefficient, used in the statistics. Both values characterize proximity, interconnection of the phenomena in question. In particular, the coefficient of proximity as the correlation coefficient, does not exceed one and is equal to it, only if signals coincide. Moreover, if  $x(t)$  and  $y(t)$  is the random ergodic processes, calibrated on the dispersion, then the correlation coefficient is formally determined by relationship/ratio (1.20). We introduced new term for the designation of this value only because in our case there is no any connection/communication with the probabilistic laws.

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1.8. Three methods of the solution of the fundamental problem of synthesis.

In Fig. 1.7 are clarified the most commonly used methods of the solution of the fundamental problem of synthesis. The first method consists in the fact that first is chosen arbitrary permissible signal  $x \in X$  and is determined best approximation to it on many

desired signals  $Y$ . In other words, as the first space is solved the problem of approximation (design) on set  $Y$ . The quality of the approximation is characterized by the distance

$$d(x, Y) = \min_{y \in Y} d(x, y) = \|x - P_Y(x)\|.$$

This distance realizes certain signal  $y_1 = P_Y(x)$ , nearest to selected  $x$  (let us recall that  $P_Y$  - operator of design to set  $Y$ ).

In order to obtain the shortest distance between regions  $X$  and  $Y$ , it suffices to further change signal  $x$ , being moved on by the curve  $X$  and monitoring distance of nearest  $y_1 \in Y$  (Fig. 1.7a).

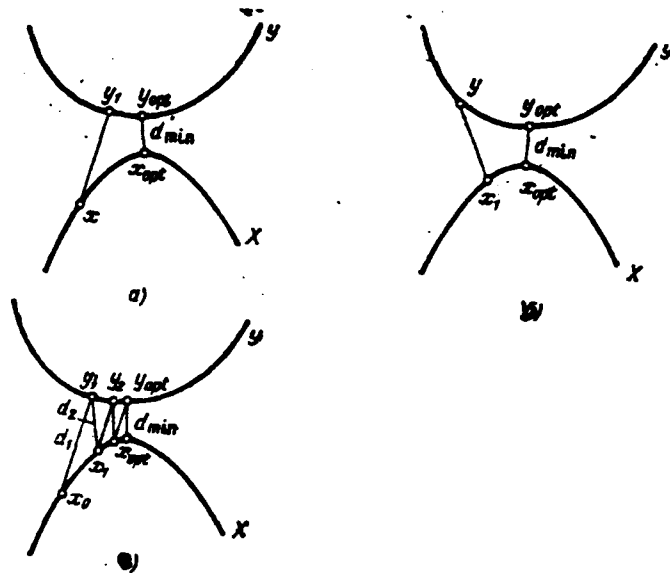


Fig. 1.7.

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As a result is determined value

$$d_{min} = \min_{\substack{y \in Y \\ x \in X}} d(x, y) = \min_{x \in X} \|x - P_Y(x)\|$$

and is located optimum signal  $x_{opt}$ , which solves the problem of synthesis in accordance with the hypothesis of proximity. unknown signal  $x_{opt}$  realizes the minimum of the functional

$$f(x) = d(x, Y) = \|x - P_Y(x)\| = \min, \quad (1.25)$$

where the minimization is produced on all  $x \in X$ . The same result is obtained during the maximization of the coefficient of proximity  $C(x,$

Y). Thus, synthesis is reduced also to the variational problem

$$f_1(x) = C(x, Y) = C(x, P_Y(x)) = \max \quad (1.26)$$

under further conditions  $x \in X$  and  $\|x\| = 1$ .

This is one of the classical problems of the calculus of variations - isoperimetric problem.

The second method differs from previous only in terms of the fact that the minimization of distance is produced in backward sequence (Fig. 1.7b). First is assigned arbitrary signal  $y \in Y$  and is found out best approximation to it on set  $X$ , i.e., is determined the distance

$$d(X, y) = \min_{x \in X} d(x, y) = \|y - P_X(y)\|.$$

Then is determined value  $d_{\min}$  via a variation in signal  $y$ . The equivalence of the first and second methods escape/ensues from the identity

$$\min_{\substack{y \in Y \\ x \in X}} d(x, y) = \min_{\substack{x \in X \\ y \in Y}} d(x, y).$$

As a result of the solution by the second method is determined not unknown signal  $x_{\text{opt}}$  and signal  $y_{\text{opt}}$  arranged/located on the shortest distance from set  $X$ . As it is clear from previous, this signal satisfies the conditions

$$f(y) = d(X, y) = \|y - P_X(y)\| = \min \quad (1.27)$$

or

$$j_1(y) = C(X, y) = C(y, P_X(y)) = \max \quad (1.28)$$

during further limitations  $y \in Y$  and  $\|y\| = 1$ . Two methods indicated, obviously, are very close.

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In the general/common/total formulation, given above, it is difficult to discover the noticeable difference between them. But difference is and it is sufficiently substantial. The fact is that the structure of the permissible signals  $x$  frequently impedes the straight/direct solution. In particular, these signals can be discrete/digital (for example, phase-keyed). The majorities of variational methods are adapted for the continuous functions. Therefore functional (1.25) or (1.26) frequently makes it possible only to establish/install, what condition satisfies unknown signal  $x_{opt}$  to come to light/detect/expose the criterion of approximation/approach, but it is possible to find the practical algorithm of synthesis, the rule of the construction of optimum signal with this approach more rarely.

Set  $Y$ , characterized by the desired property, frequently contains continuous, analytic functions. Therefore research of functional (1.27) or (1.28) can prove to be more efficient. True, after determining signal  $y_{opt}$ , arranged/located on the shortest which satisfies condition (1.27), it is then necessary to return to set  $X$  and to determine signal  $x_{opt}$ .

distance from  $y_{opt}$ . But this already the simpler problem of approximation, and its solution is comparatively simple.

Let us introduce in this connection the important for future reference concept about the generating signal. We will call signal  $y$  generating for signal  $x$ , if the latter realizes the minimum of distance (maximum of the coefficient of proximity) from  $y$  to set  $X$ . In particular, the shown in Fig. 1.7b signal  $y$  - generating for signal  $x_1$ , signal  $y_{opt}$  - generating for signal  $x_{opt}$ . It is obvious, transition/junction from the generating signal to the appropriate signal of set  $X$  gives design  $y$  on  $X$ . The method of the synthesis Fig. 1.7b can be named therefore the synthesis of the optimum generating signal with the subsequent design (approximation).

It is of interest also method successive approximations (Fig. 1.7c). With this method initially is assigned certain signal of the zero approximation  $x_0$ . Then is determined signal  $y_1 = P_Y(x_0) \in Y$ , nearest to  $x_0$ . Let the distance between these signals be  $d_1$ . Further is determined signal  $x_1 = P_X(y_1) \in X$ , nearest to  $y_1$  and arranged/located at a distance of  $d_2$  from it. Subsequently are determined signals  $y_2$  - nearest to  $x_1$ ,  $x_2$  - nearest to  $y_2$  and so forth.

Obviously, this iterative process is reduced to the successive design to sets X and Y respectively. The algorithms, which are determining approximating signals (projections on X and Y), it is frequently not difficult to establish/install, and iterative procedure proves to be sufficiently conveniently, especially during the machine calculations.

It is necessary to emphasize that this process does not always lead to shortest distance  $d_{min}$ . From previous it follows only that with the iterations is formed descending sequence of the distances

$$d_1 \geq d_2 \geq d_3 \geq \dots$$

Since this sequence is bounded below ( $d > d_{min}$ ), it converge to certain limit. But such a limit can give the local, but not global minimum of the distance between X and Y. In other words, if the curves X and Y have complicated structure, they converge at several points, then the process of successive approximations leads to the minimum of distance, but, perhaps, not smallest of the minimums. So that as a result of approximations/approaches would be obtained shortest distance  $d_{min}$ , signal of the initial approximation/approach  $x_0$  must be sufficient to close ones to  $x_{opt}$ . Finding this initial approximation/approach represents the independent problem, frequently very complicated. In more detail questions of the convergence of iterations during the successive design are examined in §1.10. Let us now point out only that many

iterative processes, used by a number of the authors for the synthesis of signals and antennas, are reduced to the described by us procedure.

#### 1.9. On the iterative methods of synthesis.

Thus, the problem of synthesis consists in general in the fact that is required to find the permissible signal  $x(t)$ , which ensures best approximation to certain desired property. This problem is reduced to the minimization of the corresponding functional - the criterion of synthesis  $f(x)$  - on a permissible multitude of signals  $X$ :  $f(x) = \min: x \in X$ . The structure of functional  $f(x)$  is determined by the desired property of the synthesized signals, and also by the fact such as kind approximation/approach to this property we attempt to obtain - minimax, quadratic, mean-degree, etc.

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The approach examined to the synthesis, based on the criterion of proximity, consists, in fact, of the special selection of functional  $f(x)$ . It is the distance between the appropriate sets in the assigned function space. Frequently this criterion makes it possible to trace the problem of minimization (i.e. synthesis) by classical variational methods and to obtain the analytical solution.

But this accomplishes not always, and we recently mentioned the iterative procedure, which makes it possible in a number of cases to obtain successive approximations to the unknown solution. Generally, one ought not to narrow the circle of the problems in question, being oriented only to the criterion of proximity and to the analytical methods of solution. Therefore let us consider briefly also the numerical minimizations, applied, in the principle, to the arbitrary ones (or almost arbitrary) functionals.

Such methods they allow being transmitted from certain initial approximation/approach  $x^{(0)}(t)$ , to find successive approximations  $x^{(1)}(t), x^{(2)}(t), \dots$ , reducing step by step functional being investigated value. In other words, is constructed minimizing sequence

$x^{(0)}(t), x^{(1)}(t), x^{(2)}(t), \dots$ , that satisfying the condition

$$f(x^{(k+1)}) \leq f(x^{(k)}).$$

The functionals  $f(x)$  in question make sense of an error in the approximation/approach, they are always positive and, therefore, bounded below:  $f(x) \geq 0$ . Therefore descending sequence of values  $f(x^{(0)}), f(x^{(1)}), \dots$  converge to certain limit, and, after fulfilling a sufficient number of spaces, it is possible arbitrary closely to approach this limiting value.

However, the majority of the iterative methods of minimization have local character. During the construction of next approximation/approach is considered the behavior of the functional being investigated only in certain low vicinity of the previous approximation/approach. Speaking in general terms, each following approximation/approach  $x^{(k+1)}(t)$  introduces only small correction into previous approximation/approach  $x^{(k)}(t)$ . Although this correction gives the decrease of functional, the process of minimization can, in the first place, flow/occur/last very slowly, and in the second place, it leads to the nearest - frequently to the local, but not global minimum. We approach the minimum, but frequently smallest of them.

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Let us examine in more detail the method of gradient - one of frequently that utilized and in a certain sense fundamental iterative minimization. As it was noted, any signal  $x(t)$  it is possible to unambiguously compare the sequences of numbers  $\{x_1, x_2, \dots, x_n, \dots\}$  - the coefficients of the generalized series of Fourier (1.4). Therefore functional  $f(x)$ , depending on signal  $x(t)$ , can be considered as the function of many variable/alternating  $f[x(t)] = f(x_1, x_2, \dots, x_n, \dots)$ . In the geometric analogy  $f(x)$  there is a surface above the hyperplane sufficiently large, strictly speaking infinite, number of coordinates (the hyperplane indicated we they treated the previously as space of

signals).

Let initial approximation/approach  $x^{(0)}(t)$  correspond to point on this hyperplane with coordinates  $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}, \dots$ . Examining the behavior of function  $f(x)$  near point  $x^{(0)}$ , it is possible to come to light/detect/expose directions, along which  $f(x)$  grows or, on the contrary, it decreases. Attempting to decrease value of  $f(x)$ , it is necessary, naturally, to be shifted from point  $x^{(0)}$  on one of the latter/last directions. Moreover in order to obtain the most significant decrease of  $f(x)$ , one should be shifted in the direction of most steepest/most abrupt descent, antigradient. The coordinates of following, the first, approximations/approaches  $x^{(1)}(t)$  in this case will compose

$$x_n^{(1)} = x_n^{(0)} - \alpha_0 \frac{\partial f}{\partial x_n} \Big|_{x=x^{(0)}}. \quad (1.29)$$

Here  $\alpha_0$  - positive constant, which is determining the length of space. After fulfilling first approximation, it is necessary to study the behavior of function  $f(x)$  near point  $x^{(1)}$  and, after determining the new direction of antigradient - direction of steepest/most abrupt descent from point  $x^{(1)}$ , to be shifted in this direction for obtaining the second approximation/approach  $x^{(2)}$ . In general, transition/junction from the  $k$  approximation/approach to the following is determined by the relationship/ratio

$$x_n^{(k+1)} = x_n^{(k)} - \alpha_k \frac{\partial f}{\partial x_n} \Big|_{x=x^{(k)}}. \quad (1.30)$$

Usefully also another interpretation of the process examined.

The extremum of function  $f(x)$ , naturally, gives the solution of system of equations

$$\frac{\partial f}{\partial x_n} = 0; \quad n = 1, 2, \dots$$

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It is not difficult to note that process (1.30) corresponds to the iterative solution of this system, it is more precise, the equivalent system of the form

$$x_n = x_n - \alpha \frac{\partial f}{\partial x_n}; \quad n = 1, 2, \dots$$

the use of sufficiently low values of  $\alpha$  frequently ensuring the convergence of iterations.

Known several varieties of gradient method, which are characterized by the rule of the selection of the length of space  $\alpha$ . In the simplest case the method descends in the version of the simple iteration when the length of space remains constant in all stages

$$\alpha_0 = \alpha_1 = \dots = \alpha.$$

In the more complicated cases the convergence is ensured only with the variable space, selected, for example, from the following considerations.

During the motion in the direction of antigradient changes  $f(x)$  occur in accordance with the one-dimensional curve

$$f = f\left(x_1 - \alpha \frac{\partial f}{\partial x_1}, \dots, x_n - \alpha \frac{\partial f}{\partial x_n}, \dots\right) = \varphi(\alpha) \quad (1.31)$$

- by the section of multidimensional function  $f(x_1, \dots, x_n, \dots)$  by the vertical plane, including the selected direction. Move one should, naturally, to the point of the minimum in this section. Therefore in order to determine the length of space, it is possible in each stage to solve the one-dimensional problem of minimization for function (1.31). The corresponding methods are examined, for example, by Wild [78]. In particular, if curve  $f(\alpha)$  allows/assumes approximation by the quadratic parabola

$$\varphi(x) \approx \varphi(0) + \varphi'(0)x + \frac{\varphi''(0)}{2}x^2,$$

the value  $\alpha_k$  is determined by the position of its apex/vertex

$$\alpha_k = -\varphi'(0)/\varphi''(0).$$

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The method of gradient is one of the numerical methods of minimization, used in similar problems. There are other analogous methods, in particular design- gradient, that considers the limitations, assigned on the permissible set of the  $X$ , the method coordinate-by-coordinate descent (relaxation), which does not require the calculation of the direction of gradient, ravine method and the method of layouts, that accelerate minimization in some important cases, etc. These methods are used extensively for the numerical solution of the diverse problems of minimization, including for the synthesis of signals [34, 53, 61, 85], and we will use them in the

series/row of complex problems. But should be specified two fundamental difficulties, connected with the use/application of such methods.

First, there does not exist the universal numerical method of minimization. The success of that, etc. of them depends substantially on the thin properties of the functional - number of local minimums, presence or absence of "ravines being investigated", their structure, etc. Similar properties not always can be traced previously, and it is expedient to test several methods, to fit method to the problem.

In the second place, the already noted local character of iterative methods does not make it possible to in general find out the global minimum of functional. In this connection the determining role plays the correct selection of initial approximation/approach during which the iterations originate from the point, located "in the zone" of the global minimum. The determination of initial approximation/approach this is the independent, frequently very complex problem for solving which are necessary analytical research or physical arguments, which make it possible to contain the structure of functional as a whole (let it be approximately, with some simplifications, but as a whole, but local, in the low vicinity of certain point).

Specifically, during finding of initial approximation/approach are exhibited the advantages of approach; basing on the criterion of proximity. We choose the space metrics of signals arbitrarily, without a strict proof. The initial problem of synthesis is substituted with this another, are changed criterion itself, minimized functional. But this replacement makes it possible to frequently contain problem as a whole, to find analytical resolution, and intuitive considerations make it possible to consider that the simplifying assumptions are not too rough. Thus, ChM signals, which implement best approximation to the assigned function of uncertainty/indeterminacy, hardly considerably differ from each other with two criteria of approximation/approach - quadratic and minimax.

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The case of quadratic approximations/approaches admits simple solution on the base of the hypothesis of the proximity (see Chapter 8), and the analytical study of minimax approximations is virtually impracticably. But here it is possible to use one of the iterative methods, using as the initial approximation/approach the result, obtained for the quadratic criterion. The corresponding refinements via iterations will lead to the global minimum of error, if two criteria indicated do not give essentially different signals. At

least, one ought not to disregard such possibility when there are no direct methods of the solution.

We will meet also with the simpler problems where the comprehensive solution can be obtained, applying either only the criterion of proximity or only iterative methods.

In the examination of gradient method, it was assumed that signal  $x(t)$  was represented by the set/dialing of its coordinates:  $x_1, x_2, \dots, x_n, \dots$ . With respect to the projection of gradient (vector) on the coordinate axes a partial derivatives  $\partial f / \partial x_n$ . Although this representation is admissible for the majority of problems, let us give the more general/common/total determination of the gradient (by first-order derivative) of functional. Let  $H$  - space of signals, and the real functional  $f(x)$  is determined for all  $x \in H$ . The differential of functional  $f(x)$  is called expression ([42], page 434) 1:

$$Df(x, h) = \frac{d}{dt} f(x + th) \Big|_{t=0} = \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t}.$$

FOOTNOTE 1. It is more precise, by Gateau differential or by weak differential. ENDFOOTNOTE.

According to the usual rules of differentiation,  $Df(x, h)$  linearly depends on  $h$ , and, since increment  $Df(x, h)$  is a scalar real value, is correct the representation

$$Df(x, h) = \left. \frac{d}{dt} f(x + th) \right|_{t=0} = \operatorname{Re}(f'(x), h). \quad (1.32)$$

Was here used the fact that in the Hilbert space the linear functional always can be presented in the form of scalar product [42]. The entering in (1.32) function (operator)  $f'(x)$ , which depends on  $x$ , but not from  $h$ , and there is derivative of functional  $f(x)$ . It is easy to see that during the coordinate-by-coordinate representation of signals (in the Euclidian space) this determination coincides with that used above.

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The general formula of gradient method (1.30) in the new designations takes the form

$$x^{(k+1)} = x^{(k)} - \alpha_k f'(x^{(k)}). \quad (1.33)$$

1.10. A projective-gradient method and its connection/communication with the criterion of proximity.

Gradient method is applied when on the permissible signals  $x(t)$  it is not placed any limitations, i.e., when set  $X$  it corresponds to entire space  $H$ . If set  $X$  is limited by further conditions, method (1.33) cannot be used, since the addition of derivative  $f'(x)$  can deduce for the limits of the permissible set and next

approximation/approach  $x^{k+1}$  will not belong to  $X$ .

However, under these conditions, common for the problems of minimization, it is possible to use, for example, the appropriate modification presented, called a projective-gradient method. During this modification next approximation/approach is constructed according to the rule

$$x^{(k+1)} = P_X[x^{(k)} - \alpha_k f'(x^{(k)})], \quad (1.34)$$

where  $P_X$  - operator of design to set  $X$ .

The projective-gradient method prescribes, thus, to do from point  $x^{(k)}$  a space on the antigradient (as in the previous method, without taking into account limitations on  $X$ ), and then to define next approximation/approach  $x^{k+1}$  as the point of set  $X$ , nearest to obtained point  $x^{(k)} - \alpha_k f'(x^{(k)})$ .

Let us look how appears method (1.34) in connection with the minimization of the distance between permitted to  $X$  and desired  $Y$  multitudes of the space of signals, in our terminology - for the fundamental problem of synthesis.

Functioning using the first method of §1.8, we fix/record first arbitrary element/cell  $x \in X$  and it is determined shortest distance of  $Y$ . Equivalent component  $x_1 \in Y$  is projection  $x$  on  $Y$ , and distance  $d(x,$

Y) comprises

$$d(x, Y) = \|x - P_Y(x)\|.$$

The least distance between X and Y is determined further by a variation in the permissible signals x. Thus, we come to the minimization of functional [40]

$$f(x) = d^2(x, Y) = \|x - P_Y(x)\|^2 = (x - P_Y(x))^\top (x - P_Y(x)) \quad (1.35)$$

on all  $x \in X$ . Computing derivative according to rule (1.32), we obtain<sup>1</sup>

$$f'(x) = 2(x - P_Y(x)), \quad (1.36)$$

FOOTNOTE 1. Functioning strictly, during the calculation of derivative should be constructed linear variety, tangent to Y at point  $y_0 = P_Y(x)$ , and, using the smallness of increment  $\Delta x$  to replace design by Y with design to the variety indicated, and then consider the known properties of the operator of design to linear subspace (see [42, page 312, 480]). ENDFOOTNOTE.

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Therefore the projective-gradient method leads to the following rule of the construction of the successive approximations:

$$x^{(k+1)} = P_X \{x^{(k)} - 2\alpha_k (x^{(k)} - P_Y(x^{(k)}))\}.$$

In the version of simple iteration with space  $\alpha_k = 1/2$  we obtain especially simple rule

$$x^{(k+1)} = P_Y \{P_X(x^{(k)})\}. \quad (1.37)$$

indicating, obviously, consecutive design to sets  $Y$  and  $X$  respectively.

Was obtained that very procedure of successive design which was proposed in §1.8 as one of the methods of the solution of the fundamental problem of the synthesis (see Fig. 1.7c). The procedure indicated corresponds, thus, to a projective-gradient method for functional (1.35) in the particular case  $\alpha_k = 1/2$ .

This conformity is important on the following reason. In §1.8 we led only the partial proof of the method of successive design. It was shown that consecutive distances between  $X$  and  $Y$  monotonically decrease, approaching certain limit. In other words, is established/installed only the convergence of process on functional  $f$ . Now, taking into account connection/communication with a projective-gradient method, it is possible to use the known conditions for its applicability (and also by some of their expansions - see below) in order to trace questions of convergence more fully, to establish/install, in what cases successive approximations  $x^{(1)}, x^{(2)}, \dots$  approach optimum signal  $x_{opt}$ , or at least they converge, so that the norm of difference  $\|x^{(k+1)} - x^{(k)}\|$  vanishes.

Let us begin from some determinations. Set  $X$  is called convex, if any two points of it can be connected by segment, without

exceeding the limits of the set indicated. In other words,

$$x = \tau x_1 + (1-\tau)x_2 \in X \text{ при } x_1, x_2 \in X \text{ и } 0 < \tau < 1. \quad (1.38)$$

Key: (1). with. (2). and.

Let  $z \in H$  the arbitrary point of space and  $x_1 = P_X(z)$  - nearest to it point of set  $X$ . Then, if  $X$  is convex, occurs the inequality

$$\operatorname{Re}(z - x_1, x - x_1) = \operatorname{Re}(z - P_X(z), x - P_X(z)) \leq 0. \quad (1.39)$$

where  $x \in X$  and, as usual, the parenthesis designate scalar product. Not stopping on the proof of this known relationship/ratio, let us note that geometrically it corresponds so that in triangle  $z, x_1, x$  the apex angle  $x_1$  must be blunt for convex set  $X$  (Fig. 1.8).

Condition (1.39) is only necessary, while condition (1.38) is also sufficient so that the set  $X$  would be convex.

Functional  $f(x)$  is called convex (downward) on  $X$ , if with  $0 < \tau < 1$  occurs the inequality

$$f(\tau x_1 + (1-\tau)x_2) \leq \tau f(x_1) + (1-\tau)f(x_2); \quad x_1, x_2 \in X \quad (1.40)$$

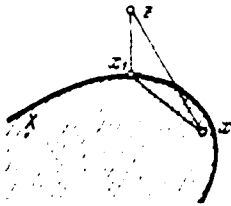


Fig. 1.8.

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For the differentiated functional the condition of convexity is equivalent also so that

$$\text{Re} (f'(x_1) - f'(x_2), x_1 - x_2) \geq 0, \quad x_1, x_2 \in X. \quad (1.41)$$

Is valid the following general/common/total theorem: convex functional has the only minimum on the locked convex set this  $X$  and minimum is reached at unique point  $x_{opt} \in X$  [40].

Further, it is accepted to indicate that function  $f'(x)$  satisfies Lipschitz condition with constant  $M$ , if

$$\|f'(x_1) - f'(x_2)\| \leq M \|x_1 - x_2\|, \quad x_1, x_2 \in X. \quad (1.42)$$

Let us now give the theorem relative to the applicability of a projective-gradient method, proved by Levitin and Polyak [40].

Let  $f(x)$  - the differentiated convex functional on convex set  $X$ , the derivative  $f'(x)$  satisfying Lipschitz condition with constant  $M$ .

Then, if the value of space  $\alpha$  is chosen in the limits

$$0 < \alpha_k < 2/M, \quad (1.43)$$

then: 1) projective-gradient method (1.34) gives monotonically descending sequence  $f(x^{(k)})$ , converging to  $f_{\min}$ :

2) sequence  $x^{(k)}$  converge to (unique) point of minimum  $x_{opt}$ , in particular  $\|x^{(k+1)} - x^{(k)}\| \rightarrow 0$ .

In the interesting us problem of the synthesis of functional  $f(x)$  it corresponds (1.35), and in order to use this theorem, it is necessary to make more precise its properties. Let set  $Y$  be convex. Then, if  $x_1, x_2$  - points of set  $X$ , which do not belong it goes without saying to  $Y$ , and  $P_Y(x_1)$  and  $P_Y(x_2)$  - their projection on  $Y$ , on the basis (1.39) we can register

$$\begin{aligned} 0 &\geq \operatorname{Re}(x_1 - P_Y(x_1), P_Y(x_2) - P_Y(x_1)) + \operatorname{Re}(x_2 - P_Y(x_2), P_Y(x_1) - \\ &\quad - P_Y(x_2)) = \operatorname{Re}(x_1 - x_2 - P_Y(x_1) + P_Y(x_2), P_Y(x_2) - P_Y(x_1)) = \\ &= \|x_1 - x_2 - P_Y(x_1) + P_Y(x_2)\|^2 - \operatorname{Re}(x_1 - x_2 - P_Y(x_1) + P_Y(x_2), x_1 - x_2). \end{aligned}$$

Or, taking into account the value of derivative (1.36),

$$\|f'(x_1) - f'(x_2)\|^2 - 2\operatorname{Re}(f'(x_1) - f'(x_2), x_1 - x_2) \leq 0$$

It is obvious, this is possible only with satisfaction of condition  $\operatorname{Re}(f'(x_1) - f'(x_2), x_1 - x_2) > 0$ , which, in turn, it indicates the convexity of functional  $f(x)$  (see (1.41)). Thus, the convexity of set  $Y$  imply the convexity of functional (1.35). Further we have

$$\begin{aligned} \|f'(x_1) - f'(x_2)\| &\leq \sqrt{2\operatorname{Re}(f'(x_1) - f'(x_2), x_1 - x_2)} \\ &\leq \sqrt{2\|f'(x_1) - f'(x_2)\|\|x_1 - x_2\|} \end{aligned}$$

Or, which is the same thing,

$$\|f'(x_1) - f'(x_2)\| \leq 2\|x_1 - x_2\|.$$

We obtained Lipschitz condition for by the derivative  $f(x)$  with constant  $M=2$ .

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Thus, if both sets  $X$  and  $Y$  are convex, then all conditions of the previous theorem are satisfied and the projective-gradient method let us use with any space  $\alpha_k < 1$ . moreover is ensured the convergence both of the distances  $f(x^{(k)})$ , and signals  $x^{(k)}$  to our (unique) optimum values. In particular, after taking  $\alpha_k = 1/2$ , let us establish the convergence of the procedure of successive design<sup>1</sup>.

FOOTNOTE <sup>1</sup>. From the results of Levitin and Polyak it follows also that when  $\alpha = 1/2$  is ensured a maximally possible speed of convergence in many important cases. ENDFOOTNOTE.

Thus, for convex sets  $X$  and  $Y$  is a comprehensive proof of the convergence of the method of successive design.

With the fulfillment of iterations they are always limited to a finite number of spaces. Therefore from a practical point of view the convergence of process frequently has smaller value than the approach

of the successive approximations  $x_1, x_2, x_3, \dots$  at the initial stage. With convex sets  $X$  and  $Y$  the iterations according to the method of successive design not only descend, but also monotonically they converge from one space to the next. Actually/really, since  $x^{(k)} = P_X(y^{(k)})$  and set  $X$  is convex, on the basis (1.39) we have

$$\begin{aligned} 0 &\geq \operatorname{Re}(y^{(k)} - x^{(k)}, x^{(k+1)} - x^{(k)}) + \operatorname{Re}(y^{(k+1)} - x^{(k+1)}, x^{(k)} - x^{(k-1)}) = \\ &= \operatorname{Re}(y^{(k)} - y^{(k+1)} - x^{(k)} + x^{(k+1)}, x^{(k+1)} - x^{(k)}) = \\ &= \operatorname{Re}(y^{(k)} - y^{(k+1)}, x^{(k+1)} - x^{(k)}) + \|x^{(k+1)} - x^{(k)}\|^2, \end{aligned}$$

i.e.

$$\begin{aligned} \|x^{(k+1)} - x^{(k)}\|^2 &\leq \operatorname{Re}(y^{(k+1)} - y^{(k)}, x^{(k+1)} - x^{(k)}) \leq \\ &\leq \|y^{(k+1)} - y^{(k)}\| \|x^{(k+1)} - x^{(k)}\|. \end{aligned}$$

Thus, we obtain

$$\|x^{(k+1)} - x^{(k)}\| \leq \|y^{(k+1)} - y^{(k)}\|.$$

Further, since  $Y$  is convex and  $y^{(k)} = P_Y(x^{(k-1)})$ , analogously we have

$$\|y^{(k+1)} - y^{(k)}\| \leq \|x^{(k)} - x^{(k-1)}\|.$$

These two inequalities indicate, obviously, the monotone approach of the consecutively/serially obtained signals<sup>2</sup>.

FOOTNOTE <sup>2</sup>. On the basis of the principle of the contracted mappings ([42] page 44) hence follows also the convergence of process to the unique point of the minimum. ENDFOOTNOTE.

But, unfortunately, with the synthesis frequently it is necessary to deal concerning the nonconvex sets. Let us assume, for example, is required to find the ChM signal, which satisfies a certain condition of optimum character. Set  $X$  includes in this case

the signals

$$x(t) = B(t) e^{j\psi(t)}, \quad (1.44)$$

characterized by only phase functions  $\psi(t)$ , which depend on the law of frequency modulation. Amplitude envelope  $B(t)$  is identical for all permissible signals (for example,  $B(t)$  can be assigned rectangular).

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This set  $X$  is not convex. Actually/really, after assuming in accordance with (1.38)

$$x(t) = \tau x_1(t) + (1-\tau)x_2(t) = B(t) [\tau e^{j\psi_1(t)} + (1-\tau)e^{j\psi_2(t)}],$$

we see that the envelope

$$|x(t)| = B(t) \sqrt{\tau^2 + (1-\tau)^2 - 2\tau(1-\tau)\cos(\psi_1(t) - \psi_2(t))}$$

differs from  $B(t)$  with any  $\tau$ , except 0 and 1, i.e., the internal points of the segment, which combines  $x_1, x_2 \in X$ , do not belong to set  $X$ . Is similar to this, frequently convexly and desired set  $Y$ .

Of course the nonconvexity at least of one of the sets brings, in general, to the presence of several minimums of the distance between them, and iterations can lead to the local, but not to global minimum. Therefore it cannot be relied on the so/such comprehensive proof of method as for the convex sets. But nevertheless we will obtain further some results, similar to previous.

Let us show first of all, that for any set  $X$ , convexly it or not, is correct the inequality, similar (1.39):

$$\operatorname{Re} \left( z - \frac{P_X(z) + x}{2}, x - P_X(z) \right) \leq 0; \quad x \in X. \quad (1.45)$$

Actually/really, since the operator of design  $P_X(z)$  assumes finding the shortest distance from  $z$  to  $X$ , with any  $x \in X$

$$\begin{aligned} 0 &\geq \|z - P_X(z)\|^2 - \|z - x\|^2 = \|z\|^2 + \|P_X(z)\|^2 - 2\operatorname{Re}(z, P_X(z)) - \\ &\quad - \|z\|^2 - \|x\|^2 + 2\operatorname{Re}(z, x) = \|P_X(z)\|^2 - \|x\|^2 + \\ &\quad + 2\operatorname{Re}(z, x - P_X(z)) = \operatorname{Re}(P_X(z) + x, P_X(z) - x) + \\ &\quad + 2\operatorname{Re}(z, x - P_X(z)) = 2\operatorname{Re} \left( z - \frac{P_X(z) + x}{2}, x - P_X(z) \right). \end{aligned}$$

Clear also that inequality (1.45) strict, if operator  $P_X(z)$  is unique, i.e., if is unique the nearest to  $z$  point of set  $X$ . Geometrically (1.45) it corresponds so that, after leading median in triangle  $z, x_1, x$ , we will obtain triangle  $z, \xi, x$  with the blunt apex angle  $\xi$  (Fig. 1.9).

Let us now demonstrate the theorem, similar (in certain part) to the theorem of Lavitin and Polyak, but not assuming the convexity of set  $X$  and functional  $f(x)$ .

Let  $f(x)$  - the differentiated functional, bounded below on set  $X$ , the derivative  $f'(x)$  satisfies lipshitz condition with constant  $M$ . Then, if the value of space  $n$  is chosen in the limits

$$0 < n_k \leq 1/M, \quad (1.46)$$

then: 1) projective-gradient method (1.34) gives monotonically decreasing convergent series  $\{x^{(k)}\}$  sequence  $x^{(k)}$  descends according to the norm of difference, i.e.,  $\|x^{(k+1)} - x^{(k)}\| \rightarrow 0$ .

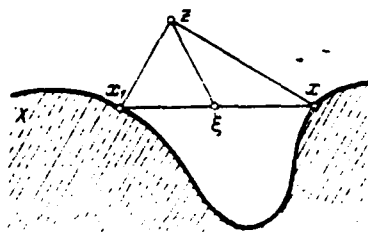


Fig. 1.9.

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The following proof repeats (with some explanations) the appropriate consideration from [40], but instead of inequality (1.39), valid for convex  $X$ , it is used (1.45). First of all, on the basis of the determination of derivative (1.32) we have

$$\begin{aligned} f(x+h) - f(x) &= \operatorname{Re} \int_0^1 (f'(x+th), h) d\tau = \operatorname{Re} (f'(x), h) + \\ &+ \operatorname{Re} \int_0^1 (f'(x+th) - f'(x), h) d\tau \leq \operatorname{Re} (f'(x), h) + \\ &+ \int_0^1 \|f'(x+th) - f'(x)\| \|h\| d\tau \leq \operatorname{Re} (f'(x), h) + \\ &+ \int_0^1 M\tau \|h\|^2 d\tau = \operatorname{Re} (f'(x), h) + \frac{M}{2} \|h\|^2. \end{aligned}$$

moreover is here used also Lipschitz condition (1.42).

Assuming/setting  $x = x^{(k)}$ ,  $h = x^{(k+1)} - x^{(k)}$ , we have in particular

$$\begin{aligned} f(x^{(k+1)}) - f(x^{(k)}) &\leq \operatorname{Re} (f'(x^{(k)}), x^{(k+1)} - x^{(k)}) + \\ &+ \frac{M}{2} \|x^{(k+1)} - x^{(k)}\|^2. \end{aligned}$$

First term of right side we convert as follows:

$$\begin{aligned} \operatorname{Re} (f'(x^{(k)}), x^{(k+1)} - x^{(k)}) &= \operatorname{Re} (-j'(x^{(k)}), x^{(k)} - x^{(k+1)}) = \\ &= \frac{1}{\alpha_k} \operatorname{Re} (x^{(k)} - \alpha_k j'(x^{(k)}) - \frac{x^{(k+1)} + x^{(k)}}{2} + \frac{x^{(k+1)} - x^{(k)}}{2}, \\ x^{(k)} - x^{(k+1)}) &= -\frac{1}{2\alpha_k} \|x^{(k+1)} - x^{(k)}\|^2 + \frac{1}{\alpha_k} \operatorname{Re} (x^{(k)} - \alpha_k j'(x^{(k)}) - \\ &- \frac{x^{(k+1)} + x^{(k)}}{2}, x^{(k)} - x^{(k+1)}). \end{aligned}$$

According to the algorithm of the construction of approximations/approaches (1.34)  $x^{(k+1)} = P_X(x^{(k)} - \alpha_k f(x^{(k)}))$  and the latter/last component/term/addend is negative by the force of inequality (1.45): as a result

$$\begin{aligned} f(x^{(k+1)}) - f(x^{(k)}) &\leq -\frac{1}{2} \left( \frac{1}{\alpha_k} - M \right) \|x^{(k+1)} - x^{(k)}\|^2 = \\ &= -\varepsilon \|x^{(k+1)} - x^{(k)}\|^2. \end{aligned}$$

If space  $\alpha_k$  is chosen according to (1.46), value  $\varepsilon$  is positive, and, thus, sequence  $f(x^{(k)})$  monotonically decreases. Since functional  $f(x)$  is bounded below, this sequence descends. Finally, we obtained also

$$\|x^{(k+1)} - x^{(k)}\|^2 \leq \frac{1}{\varepsilon} (f(x^{(k)}) - f(x^{(k+1)})).$$

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Consequently, the norm of difference  $\|x^{(k+1)} - x^{(k)}\|$  vanishes by the force of the convergence of sequence  $f(x^{(k)})$ . Theorem is proved.

From the obtained results are clear fundamental differences under applicability conditions for a projective-gradient method for the cases of presence and absence of convexity (set  $X$  and functional  $f(x)$ ). First of all, in the absence of convexity is not guaranteed convergence to the global (unique) minimum and, in the second place, upper bound for space  $\alpha_k$  must be reduced doubly [see conditions (1.43) and (1.46)].

Returning to the problem of the applicability of the method of

successive design for the fundamental problem of synthesis, we will use latter/last theorem for functional (1.35) with space  $\alpha_k = 1/2$ . From (1.46) it follows that with the nonconvex set the X convergence of method (according to the norm of difference and in the functional) is ensured, if  $M > 2$ , i.e., upon consideration (1.36) and (1.42), if

$$\|x_1 - x_2 - P_Y(x_1) + P_Y(x_2)\| < \|x_1 - x_2\|; x_1, x_2 \in X. \quad (1.47)$$

In the fundamental problem of the synthesis of set X and Y it is possible to vary by roles. Therefore the convergence indicated is ensured also with the nonconvex set Y, if

$$\|y_1 - y_2 - P_X(y_1) + P_X(y_2)\| < \|y_1 - y_2\|; y_1, y_2 \in Y. \quad (1.48)$$

These conditions (is sufficient the fulfillment of any of them) set some further limitations on Y or X respectively<sup>1</sup>.

FOOTNOTE <sup>1</sup>. Let us emphasize that for the convergence of the distances between by X and Y of the conditions indicated it is not required - this convergence was independently proved in §1.8. Inequalities (1.47) or (1.48) ensure also the convergence of approximations/approaches according to the norm of difference.

ENDFOOTNOTE.

It is possible to check that these conditions are satisfied, if for set Y or X is correct inequality (1.39). So that is sufficient the convexity only of one of the sets.

Let us look how appears condition (1.48) in connection with set of ChM signals. Let

$$y_1(t) = A_1(t) e^{j\phi_1(t)} \text{ и } y_2(t) = A_2(t) e^{j\phi_2(t)}$$

- signals with arbitrary amplitudes and phases. It is possible to show that the design of signal  $y(t)$  to the set of ChM signals (1.44) corresponds to adding of its phase  $\phi(t)$  to the assigned amplitude  $B(t)$  (see Chapter 8), so that

$$P_X(p_1) = B(t) e^{j\phi_1(t)}, P_X(p_2) = B(t) e^{j\phi_2(t)}.$$

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In space  $L^2$  condition (1.48) takes the form

$$\int |(A_1 - B) e^{j\phi_1} - (A_2 - B) e^{j\phi_2}|^2 dt < \int |A_1 e^{j\phi_1} - A_2 e^{j\phi_2}|^2 dt.$$

or, after simple conversions,

$$\int B(t) [B(t) - A_1(t) - A_2(t)] \sin^2 \frac{\phi_1(t) - \phi_2(t)}{2} dt < 0.$$

If we do not set any limitations on the phases  $\phi_1(t)$  and  $\phi_2(t)$ , we will obtain the sufficient condition

$$A_1(t) + A_2(t) > B(t), \quad (1.49)$$

which must be implemented for all  $t$  with which  $B(t) \neq 0$ . Even more rigid (also sufficient) condition is simultaneous fulfilling of two inequalities

$$A_1(t) > \frac{1}{2} B(t), A_2(t) > \frac{1}{2} B(t). \quad (1.50)$$

of those indicating, it is obvious, that the envelopes  $A_1(t)$  and  $A_2(t)$  must not differ more than doubly, from assigned envelope  $B(t)$ .

Even latter/last condition is not especially limiting. It indicates that the initial approximation/approach must ensure certain (apparently, too not high) quality of approximation, so that the desired signal too would not differ from those permitted. Then, applying successive design, we obtain the necessary refinements.

Let us note that if operators  $P_X(y)$  and  $P_Y(x)$  are unique, i.e., to each point  $y_i \in Y$  corresponds unique nearest to it point  $x_i = P_X(y_i) \in X$  and it is analogous for set  $Y$ , then entire process of iterations is completely determined by the initial approximation/approach  $x_0$ . In particular, saturation signal, if it exists, depends only on the selection of initial approximation/approach. The uniqueness of approximations/approaches occurs for the majority of problems.

## Chapter 2.

## SIGNALS WITH MAXIMUM SELECTIVITY.

In the communicating systems and radar it is frequently expedient to apply the signals of the final duration, most concentrated possible in the narrow frequency band. This makes it possible to efficiently use a frequency range of connection/communication, raising the selectivity of reception/procedure and decreasing the level of interferences. In connection with radar this problem appears in the Doppler devices/equipment when there is a set/dialing of narrow-band filters for the control/checking of target speed. The maximum concentration of signal in the band of filter leads to the decrease of remainders/residues in the adjacent channels, making it possible to raise accuracy and resolution in the speed. In both the cases it is assumed that the duration of signal is limited, so that discussion deals with the possible contraction of the spectrum for the assigned duration<sup>1</sup>.

FOOTNOTE 1. For pulse-coherent systems has in mind the structure of the envelope of burst of pulses and, correspondingly, the structure of the spectrum in vicinity of one of the harmonics of repetition

frequency. ENDFOOTNOTE.

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Analogous problem appears in ranging of the targets (but not speed), when with the fixed/recorded frequency band is required to maximally concentrate signal in of the assigned to the interval time. Because of symmetry of straight/direct and inverse transformations of Fourier these problems are equivalent.

Thus, the synthesis of signals with the fixed period of time (frequency band), maximally crowded in the assigned band frequencies (duration), is of known practical interest. These signals can be named signals with the maximum selectivity in the frequency or on the time respectively.

The properties of similar signals are studied sufficiently fully. In the initial setting this question raises even to the uncertainty principle of Heisenberg-Weyl, it is more precise, to his interpretation in the application/appendix to the vibration theory. The appropriate problem of the synthesis of signals for the first time formulated by Chalk in 1950 [17, 30]. Its new solution gave Gurevich in 1956 [22, 23]. The most complete results were obtained in 1961 of Landau and by Pollack [43] whose research has special

importance for this work, since it is based on the treatment, close to cur.

In this chapter it is not contained any new results. Main target lies in the fact that to show the possibility of the synthesis of signals with the maximum selectivity on the base of the criterion of proximity, to confirm the method of synthesis, utilized further in the new problems.

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2.1. Use/application of criterion of proximity in space with quadratic metric.

Thus, let us assume it is necessary to find the signal of duration  $T$ , maximally concentrated in the assigned band of frequencies  $(-\Omega, \Omega)$ . The permissible set  $X$  contains in this case the signals, finite in interval of  $T$ , i.e., having the assigned duration:

Key: (1) with.

$$x(t) = \begin{cases} x^{(1)}(t) & \text{при } -T/2 < t < T/2; \\ 0 & \text{при } |t| > T/2. \end{cases} \quad (2.1)$$

Furthermore, we normalize the permissible signals on the energy:

$$\|x\|^2 = \int_{-T/2}^{T/2} |x(t)|^2 dt = 1. \quad (2.2)$$

We attempt to obtain the signal, maximally concentrated in the

assigned frequency band. Therefore the desired property (not feasible accurately on set  $X$ ) consists of the finiteness of the spectrum, and set  $Y$  includes the signals whose spectra are finite in interval  $(-\Omega, \Omega)$ :

$$\tilde{y}(\omega) = \begin{cases} \tilde{y}^{(1)}(\omega) & \text{при } -\Omega < \omega < \Omega; \\ 0 & \text{при } |\omega| > \Omega. \end{cases} \quad (2.3)$$

Key: (1) with.

and it is also calibrated on the energy

$$\|y\|^2 = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\tilde{y}(\omega)|^2 d\omega = 1. \quad (2.4)$$

According to the hypothesis of proximity optimum signal  $x_{opt} \in X$  must be placed at shortest distance  $d_{min}$  from set  $Y$ . For finding this signal, functioning using the first method (§1.8), let us fix temporarily arbitrary permissible signal  $x \in X$  and let us determine first nearest to it signal  $y_1 = P_Y(x) \in Y$ . For this it is necessary to maximize the coefficient of proximity (1.21)

$$C(x, y) = \operatorname{Re} \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \tilde{x}(\omega) \tilde{y}^*(\omega) d\omega. \quad (2.5)$$

by selecting spectrum  $y(\omega)$  during limitation (2.4). Let us note that the final limits in (2.4) and (2.5) are caused by the finiteness of spectrum (2.3).

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Similar problems frequently can be solved with the help of Schwarz-Buniakowski's inequality. Applying that indicated inequality

to (2.5) and taking into account (2.4), we obtain

$$C^2(x, y) \leq \left| \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \tilde{x}(\omega) \tilde{y}^*(\omega) d\omega \right|^2 \leq \\ \leq \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\tilde{x}(\omega)|^2 d\omega \cdot \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\tilde{y}(\omega)|^2 d\omega = \|x\|_2^2, \quad (2.6)$$

where

$$\|x\|_2^2 = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\tilde{x}(\omega)|^2 d\omega \quad (2.7)$$

- is energy of signal  $x(t)$ , included in the band  $-\Omega < \omega < \Omega$ . Let us emphasize that we maximize the coefficient of proximity, selecting signal  $y$ , but, as follows from (2.6) and (2.7), this coefficient is limited by the value, which depends on signal  $x$ , but not from  $y$ . Therefore the coefficient of proximity will achieve maximum (on  $y$ ) value, if both inequalities in (2.6) become equalities. For this is necessary the proportionality of functions  $\tilde{y}(\omega)$  and  $\tilde{x}(\omega)$ :

$$\tilde{y}(\omega) = \gamma \tilde{x}(\omega); \quad -\Omega < \omega < \Omega.$$

since only in this case is reached the equality in the relationship/ratio of a Schwarz-Buniakowski. Proportionality factor it is not difficult to find from the condition for standardization

(2.4):

$$\|y_1\|^2 = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\tilde{y}_1(\omega)|^2 d\omega = \frac{\gamma^2}{2\pi} \int_{-\Omega}^{\Omega} |\tilde{x}(\omega)|^2 d\omega = \gamma^2 \|x\|_2^2 = 1.$$

Therefore  $\gamma = 1/\|x\|_2$ . Finally we can register:

$K=y$ : (1) with.

$$\tilde{y}_1(\omega) = \begin{cases} \frac{1}{\|x\|_2} \tilde{x}(\omega) & \text{if } -\Omega < \omega < \Omega, \\ 0 & \text{if } |\omega| > \Omega \end{cases} \quad (2.8)$$

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After substituting this spectrum in (2.5), we are convinced, that the coefficient of proximity actually/really attains the greatest value, determined by inequality (2.6):

$$C(x, Y) = \operatorname{Re} \frac{1}{2\pi \|x\|_2} \int_{-\Omega}^{\Omega} \tilde{x}(\omega) \tilde{x}^*(\omega) d\omega = \|x\|_2.$$

Now in order to find the shortest distance between  $X$  and  $Y$ , it is sufficient varying signal  $x(t)$ , to determine maximum also on  $x$ :

$$C(X, Y) = \max_{x \in X} C(x, Y) = \max_{x \in X} \|x\|_2.$$

From (2.7) it is clear that value  $\|x\|_2^2$  is partial energy of signal  $x(t)$ , included in the band  $-\Omega < \omega < \Omega$ . Therefore  $x_{opt}$  is the signal of the assigned duration, which contains the maximum part of its energy in the assigned frequency band.

It is obvious, this corresponds also to the minimum of energy out of the band indicated how is ensured high selectivity in the frequency. Specifically, this "energy" criterion of optimum character is used in the works of Chalk [17] and Gurevich [22].

We began the solution of the problem of synthesis, without defining concretely the condition of optimum character. It was necessary in accordance with the hypothesis of proximity to only find signal  $x_{opt}$ , arranged/located on the minimum distance from set  $Y$ . In this case was used space  $L^2$  with quadratic metric (1.6). Instead of

the condition of optimum character we assigned space metrics and, furthermore, determined sets  $X$  and  $Y$  in accordance with the content of problem. The solution however satisfies the commonly used criterion of optimum character, which has clear physical treatment. This confirms the applicability of the hypothesis of proximity to the case in question.

Analogous condition satisfies signal  $y_{opt}$  that possessing maximum selectivity on the time, but not in the frequency:  $y_{opt}$  is a signal with the assigned width of the spectrum, that contains the maximum part of its energy in the assigned duration.

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## 2.2. Maximization of partial energy.

Partial energy of signal  $x(t)$ , included in the band  $(-\Omega, \Omega)$ , can be counted as follows:

$$\begin{aligned} E_2 = \|x\|_2^2 &= \frac{1}{2\pi} \int_{-\Omega}^{\Omega} |\tilde{x}(\omega)|^2 d\omega = \\ &= \frac{1}{2\pi} \int_{-\Omega}^{\Omega} d\omega \int_{-T/2}^{T/2} x^*(t) e^{j\omega t} dt \int_{-T/2}^{T/2} x(t') e^{-j\omega t'} dt' = \\ &= \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} x(t') x^*(t) G_2(t, t') dt dt', \end{aligned} \quad (2.9)$$

in this case

$$G_2(t, t') = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{j\omega(t-t')} d\omega = \frac{\sin \Omega(t-t')}{\pi(t-t')}. \quad (2.10)$$

According to said, signal with the maximum selectivity in the frequency converts into the maximum value (2.9) under further condition (2.2). Both the maximized value and the further normalization condition are quadratic functionals relative to the unknown signal; therefore it is possible to use the well known receptions/procedures of the calculus of variations in order to complete the solution.

It is not difficult to show that the unknown signal satisfies the equation

$$\int_{-T/2}^{T/2} x(t') G_p(t, t') dt' = \lambda x(t), \quad (2.11)$$

which it has solutions only at some values  $\lambda = \lambda_n$  — eigenvalues. The corresponding solutions, signals  $x_n(t)$  are the eigenfunctions of equation (2.11).

Eigenvalues  $\lambda_n$  allow/assume simple interpretation. Let for certain  $\lambda_n$  equation (2.11) be satisfied by function  $x_n(t)$ . Let us multiply left and right side of the equation on  $x_n^*(t)$  and let us integrate in the interval  $(-T/2, T/2)$ . We will obtain

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$$\int_{-T/2}^{T/2} \int_{-T/2}^{T/2} x_n(t') x_n^*(t) G_p(t, t') dt dt' = \lambda_n \int_{-T/2}^{T/2} |x_n(t)|^2 dt.$$

In accordance with (2.9) the integral to the left is partial energy  $E_p$  containing in the band  $(-\Omega, \Omega)$ . Integral in the right side is

equal to one according to the condition for standardization (2.2). Therefore eigenvalue  $\lambda_n$  is numerically equal to partial energy  $E_n$ :  $\lambda_n = E_n$ .

We should determine the signal, which maximizes partial energy, i.e., corresponding to maximum eigenvalue  $\lambda_{\max} = \lambda_0$ . Thus, signal with the maximum selectivity of the frequency is the eigenfunction of equation (2.11), which corresponds to maximum eigenvalue  $\lambda_0$ .

Analogous considerations show that the signal with the maximum selectivity on the time has a spectrum  $\tilde{s}(\omega)$ , which is the eigenfunction of the equation

$$\int_{-\Omega}^{\Omega} \tilde{s}(\omega') G_T(\omega, \omega') d\omega' = \lambda \tilde{s}(\omega) \quad (2.12)$$

and it answers maximum eigenvalue  $\lambda_0$ ; kernel  $G_T(\omega, \omega')$  is determined by the formula

$$G_T(\omega, \omega') = \frac{\sin \frac{T}{2}(\omega - \omega')}{\pi(\omega - \omega')}. \quad (2.13)$$

Integral equations (2.11) and (2.12) easily are reduced to the known equations for the sphercidal functions (see the appendix)

$$\begin{aligned} x_{opt}(t) &= \psi_0(2t/T) \quad \text{при } |t| < T/2, \\ \tilde{y}_{opt}(\omega) &= \psi_0(\omega/\Omega) \quad \text{при } |\omega| < \Omega. \end{aligned}$$

Key: with.

Sphercidal functions are studied sufficiently fully, and we will point out their below properties, which characterize signals with the

maximum selectivity.

Value  $\lambda_0$  is the portion of energies of signal, included in the assigned band (with the selectivity in the frequency) or in the assigned time interval (with the selectivity on the time). Partial energy  $\lambda_0$  depends on parameter  $c = \Omega T/2$ . This dependence is shown in Fig. 2.1.

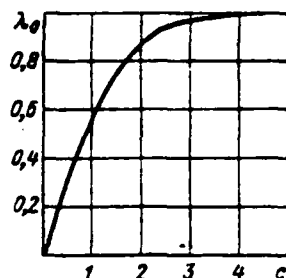


Fig. 2.1.

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For the high values of  $c$  is valid asymptotic formula [65, 68]

$$1 - \lambda_0 = 4 \pi c^{1/2} e^{-2c} \left[ 1 - \frac{3}{32c} + O(c^{-2}) \right],$$

showing that extraband energy very rapidly decreases with the expansion of band. Let us emphasize that in view of the previous conclusion/output not one signal of the assigned duration can have larger energy in the assigned band.

Signals with the maximum selectivity are depicted in Fig. 2.2. With increase in  $c$  they acquire explicit bell-shaped character. At very high values of  $c$  the signals with the maximum selectivity approach gaussian ones [67]<sup>1</sup>.

$$\psi_0(\xi) \sim e^{-\frac{c}{2} \xi^2}. \quad (2.14)$$

FOOTNOTE 1. Formula (2.14) is valid in the middle part of the signal, with the low ones  $\xi$ . For  $\xi=1$  is a noticeable difference from the gaussian curve.

Figure 2.3 illustrates the behavior of spheroidal functions on an infinite interval. These curves portray graphs of spectral density (for signals with maximum frequency selectivity) or time graphs (for signals with maximum time selectivity).

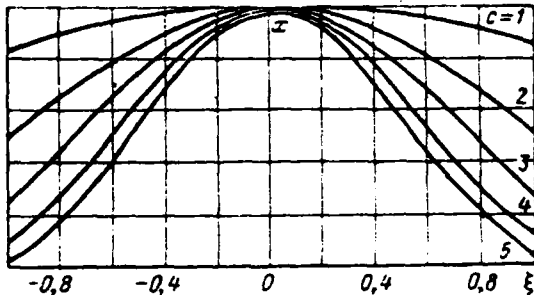


Fig. 2.2.

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For the signals with the maximum selectivity in the frequency argument  $\eta$  corresponds to dimensionless frequency ( $\eta = \omega T/2 = c\omega/\Omega$ ), while for the signals with the maximum selectivity on the time - ( $\eta = \Omega t = 2ct/T$ ) dimensionless time

### 2.3. Use/application of criterion of proximity in space with Chebyshev metric.

The synthesis of signals with the maximum selectivity is one of a few problems whose solution succeeds in obtaining not only in space  $L^2$ , but also in space  $C$  with the Chebyshev metric. This solution is of interest from two point of view. First, this example shows that the criterion of proximity can successfully be used for obtaining not only the quadratic, but also the minmax approximations. In the

second place, having solutions in two different metrics we obtain the possibility to compare them and to be convinced, at least based on particular example, that the selection of space metrics frequently does not lead to the qualitatively different results.

Examining for the concreteness signals with the maximum selectivity in the frequency, let us designate, as earlier, through a  $X$  multitude of signals, limited in the duration, and through  $Y$  - many signals, limited on the band.

We will solve the problem of synthesis in the space of the spectra, but in contrast to previous let us introduce in this space Chebyshev metric. In other words, the distance between signals  $x(t)$  and  $y(t)$  let us determine by the relationship/ratio

$$d(x, y) = \max_{\omega} |\tilde{x}(\omega) - \tilde{y}(\omega)|. \quad (2.15)$$

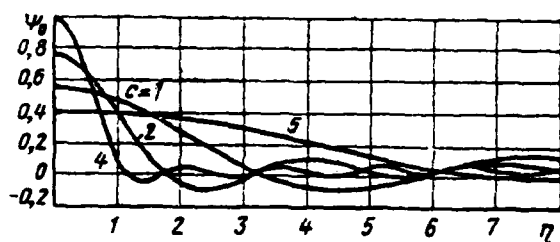


Fig. 2.3.

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According to the hypothesis of proximity the signal with maximum selectivity  $x_{opt}$  realizes the minimum distance between  $X$  and  $Y$ , i.e.

$$d_{min} = \min_{\substack{x \in X \\ y \in Y}} d(x, y).$$

Functioning analogous with previous, let us fix first arbitrary signal  $x \in X$  and it is determined shortest distance of set  $Y$ :

$$d(x, Y) = \min_{y \in Y} d(x, y) = \min_{y \in Y} \max_{\omega} |\tilde{x}(\omega) - \tilde{y}(\omega)|.$$

According to the condition, set  $Y$  contains the signals whose spectra  $\tilde{y}(\omega)$  are arbitrary in the band  $(-\Omega, \Omega)$  and are equal to zero out of this band. It is obvious, among these signals it will be located by such, for which in the band  $(-\Omega, \Omega)$  function  $\tilde{y}(\omega)$  coincides with selected  $\tilde{x}(\omega)$ . Consequently, distance  $d(x, Y)$  is determined by values of  $\tilde{x}(\omega)$  out of the band indicated, i.e., in the region where  $\tilde{y}(\omega) \equiv 0$ . Thus, we obtain

$$d(x, Y) = \max_{|\omega| > \Omega} |\tilde{x}(\omega)|. \quad (2.16)$$

Signal  $x(\tau)$  is limited by duration; therefore its spectrum  $\tilde{x}(\omega)$

is analytic function, different from zero in any frequency interval. Consequently, value (2.16) is positive, and the condition of optimum character takes the form

$$\max_{\omega \in \Omega} \tilde{A}(\omega) = \min_{\omega \in \Omega} \tilde{A}(\omega) \quad (2.17)$$

Here minimization is produced on all  $\omega \in \Omega$ , the signal, which satisfies this condition, realizes minimum distance  $d_{\min}$  in the space in question. As usual, to this signal is superimposed also the condition for standardization  $\|x\| = 1$ . With metric (2.15) this condition fixes/records the maximum value of the spectral density

$$\max_{\omega \in \Omega} \tilde{A}(\omega) = 1. \quad (2.18)$$

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#### 2.4. Dolph-Chebyshev type signals.

Relationships/ratios (2.17)-(2.18) formulate the minimax (uniform) condition for the best approximation. Thus, the use/application of a hypothesis of proximity in space  $C$  brought us to another criterion of approximation/approach, than in space  $L^2$ , but also to one of the commonly used criteria.

The solution of this problem does not succeed in obtaining by so direct method as with the quadratic approximations/approaches, but it is possible to use the following approach. Let us decompose duration  $T$  in the low sections  $\delta = T/2n$  and we will consider that function  $x(t)$

is constant within each section (it is equal to  $x_k$ ). In other words, we substitute the continuous function  $x(t)$  of corresponding stepped curve (Fig. 2.4). For the fixed/recorded number of steps/stages  $2n$  it is possible to determine the optimum function  $x(\omega)$ , with which is satisfied the condition for best approximation (2.17), and further, passing to limit of  $n \rightarrow \infty$ , to obtain the unknown continuous signal.

We will not dwell during the solution indicated. Initially it was obtained in the theory of antennas [27, 47, 63]. The corresponding antennas have the minimum level of the greatest minor lobe of diagram with the assigned width of principal ray and are called Dolph-Chebyshev (since Dolph for the first time traced such antennas, and the solution is based on the properties of Chebyshev polynomials). This name is used also for the examined/considered by us signals with analogous properties [39]. Detailed unpackings/facings, which lead to relationships/ratios indicated below, are, for example in [7].

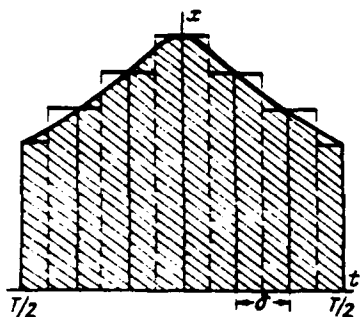


Fig. 2.4.

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The spectrum of a Dolph-Chebyshev signal is determined by one of the following formulas:

$$\tilde{x}(\omega) = \frac{1}{M} \cos c \sqrt{\frac{\omega^2}{\Omega^2} - 1} \quad (2.19)$$

or

$$\tilde{x}(\omega) = \frac{1}{M-1} \left[ \cos c \sqrt{\frac{\omega^2}{\Omega^2} - 1} - \cos c \frac{\omega}{\Omega} \right] \quad (2.20)$$

moreover (2.19) corresponds to the strictly optimum, but unrealizable signal, and (2.20) - quasi-optimal, realized. Value  $c = \Omega T/2$  characterizes, as earlier, the product of duration to the band, while value  $M > 1$  measures the residual/remnant level of the spectrum out of the assigned band:

$$\max_{|\omega| > \Omega} |\tilde{x}(\omega)| = 1/M, \quad (2.21)$$

in this case condition (2.18) is assumed to be that carried out and

$$M = \text{ch } c = \text{ch } \Omega T/2. \quad (2.22)$$

Let us note also that the major lobe/lug of the spectrum corresponds to region  $|\omega| < \Omega$ , where the trigonometric cosine in (2.19) passes into the hyperbolic.

The Dolph-Chebyshev signal  $x(t)$  is determined by Fourier transform from (2.20). This it gives

$$x(t) = \begin{cases} \frac{I_1(c\sqrt{1-\xi^2})}{I_1(c)\sqrt{1-\xi^2}} \frac{1}{\text{ном}} & -1 < \xi < 1; \\ 0 & \text{ном } |\xi| > 1. \end{cases} \quad (2.23)$$

Key: (1)  
with.

Here  $\xi = 2t/T$  - dimensionless time,  $I_1$  - the modified Bessel function. In the case (2.19) the signal has further surges on the edges (with  $t = \pm T/2$ ) of the type of delta-function. These surges cannot be realized virtually, since the pulse power of transmitter is always limited.

Fig. 2.5 shows the optimum (are more precise, quasi-optimal) signals, constructed according to formula (2.23). The parameter is the level of remainders/residues  $M$ , which depends, as it was noted, from the product  $\Omega T$ . With the low remainders/residues the signals have bell-shaped character. This is confirmed by the asymptotic formula

$$x(t) \approx e^{-c^2 \xi^2 / 2},$$

of that obtained from (2.23) with  $c \gg 1$  and  $\xi \ll 1$ .

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After comparing this formula with (2.14), we see that in the asymptotic approximation/approach the optimum signals, obtained on the base of two criteria of approximation/approach - quadratic and uniform - coincide. The more complete comparison of these signals is given in Fig. 2.6, where for value of  $c=4$  are constructed the corresponding graphs. In the same figure there is asymptotic Gaussian curve. The signals, satisfying two criteria indicated, are sufficiently close. The greatest differences are near the edges of impulse/momentum/pulse, with  $\epsilon \sim 1$ . The value of jump on the edges defines, as is known, amplitude and the speed of the decrease of the spectrum with the large ones  $\omega$ . Therefore certain disagreement of curves is caused by different requirements for the structure of the spectrum out of the assigned band.

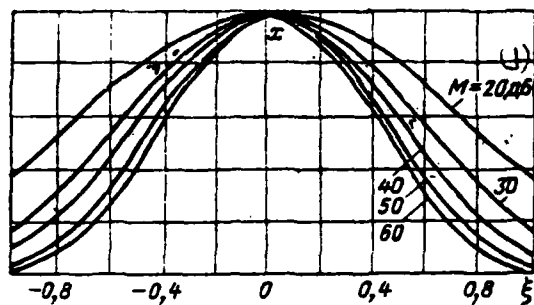


Fig. 2.5.

Key: (1) dB.

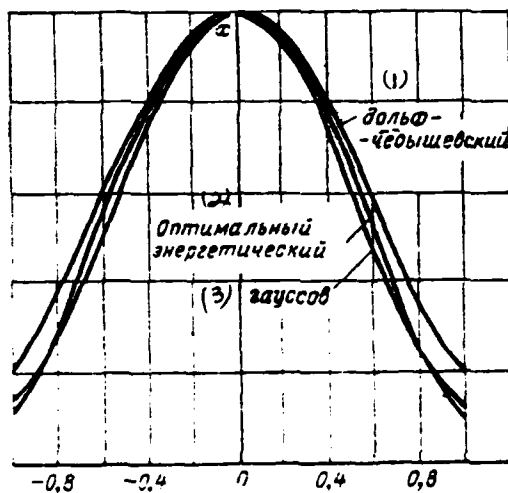


Fig. 2.6.

Key: (1). Dolph-Chebyshev. (2). Optimum energy. (3). Gauss.

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2.5. Optimum autocorrelation functions.

The synthesis of signals with the maximum selectivity on the time has close analogy with the selection of optimum autocorrelation functions. The autocorrelation function

$$\begin{aligned}
 R(t) &= \int_{-\infty}^{\infty} s(t' + t/2) s^*(t' - t/2) dt' = \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{s}(\omega)|^2 e^{j\omega t} d\omega
 \end{aligned} \quad (2.24)$$

is formed at the output of the receiver, matched with the signal.

This is an apparatus function in rangings. One of the main problems of the synthesis of signals is determination  $s(t)$ , for which  $R(t)$  has the assigned form.

This problem is in detail examined in chapter 4, but the already obtained results make it possible to indicate the optimum structure  $R(t)$ , assuring best permission/resolution under some conditions.

Fundamental requirement consists in this case in the maximum concentration  $R(t)$  in the sufficiently low interval of time near  $t=0$ . Widening the spectrum of signal, it is possible to arbitrarily decrease the duration of autocorrelation function. Therefore during finding of the optimum form of  $R(t)$  it is expedient to bound the width of the spectrum by the assigned band  $(-\Omega, \Omega)$  and to seek  $P(\omega)$

most crowded in the assigned time interval  $(-T/2, T/2)$ . We come to the problem, analogous to the synthesis of signals with the maximum selectivity in the time.

Many desired functions  $Y$  include, as earlier, all functions, finite in the interval  $(-T/2, T/2)$ . The permissible set  $X$  contains the functions, limited on the frequency band. However, these functions must be subordinated also to further condition.

As it is clear from (2.24), Fourier transform from  $R(t)$  takes the form

$$R(\omega) = S(\omega)^2.$$

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Consequently, set  $X$  contains the functions whose spectrum is positive (more precisely it is non-negative) at all values  $\omega$ . This further condition differs the problems indicated<sup>1</sup>.

FOOTNOTE is one additional difference, connected with the standardization which for the signals with the maximum selectivity and for the correlation functions, strictly speaking, is different. For greater detail, see chapter 4 (note on page 115). ENDFOOTNOTE.

The problem about the signals with the maximum selectivity was

solved by us without this limitation, i.e., on a wider multitude of the permissible functions. However, as follows of that obtained earlier results, the spectrum of optimum signals proved to be positive. Consequently, the formulas of present chapter determine also optimum autocorrelation functions. With the quadratic criterion of approximation/approach we have

$$\begin{aligned} R(\omega) &= |s(\omega)|^2 = \psi_0(\xi), \quad |\xi| \leq 1; \\ R(t) &= \psi_0(\eta). \end{aligned} \quad (2.25)$$

In the case of minimax approximations respectively it is obtained

$$\begin{aligned} \tilde{R}(\omega) &= |s(\omega)|^2 = \frac{I_1(c\sqrt{1-\xi^2})}{I_1(c)\sqrt{1-\xi^2}}, \quad |\xi| \leq 1; \quad (2.26) \\ R(t) &= \frac{1}{M-1} [\cos c\sqrt{\eta^2-1} - \cos c\eta]. \end{aligned}$$

In these formulas  $\xi = \omega/\Omega$  - dimensionless frequency,  $\eta = \Omega t = 2ct/T$  - dimensionless time. Let us note that parameter  $c$  enters also into the solution, based on the quadratic criterion, that it is not clearly indicated in (2.25).

Summarizing the fundamental results of this chapter, let us note that the use/application of a criterion of proximity to the determination of signals with the maximum selectivity leads to the results, obtained by other previously methods. The solution of this problem in space  $L^2$  reveals/detects the signals, the maximum part of energy of which is concentrated in the assigned frequency band or in the assigned time interval. On this condition of optimum character are based the works of Chalk, Gurevich and some others.

The signals with the maximum selectivity, which correspond to the condition indicated, are described by the spheroidal functions whose properties are of interest also for other problems of the theory of the signals (see below).

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The hypothesis of proximity can be used to the problem in question and in the space with the Chebyshev metric. In this case we come to the minimax criterion of approximation/approach, and the solution give function of the type of Dolph-Chebyshev, known from the theory of antennas.

The results, obtained for the signals with the maximum selectivity, are applicable also to the optimum autocorrelation functions, feasible with the limited frequency band.

## Chapter 3.

## OPTIMUM WEIGHT PROCESSING.

The practical realization of signals with the maximum selectivity meets with serious difficulties. Signal with the maximum selectivity in the frequency has bell-shaped envelope and in terms of the fact to the larger degree it differs from rectangular, the greater the product  $BT$  and the less extraband energy.

For the realization of similar signals it is necessary that in the transmitter would be implemented a deep amplitude modulation. This it gives, at least, to the considerable insufficient utilization of transmitter according to the average/mean power (energy of signal), since the pulse power is always limited. Furthermore, powerful/thick generators of SVCh work in the mode/conditions of a deep saturation. They are not adapted for amplitude modulation; sufficiently precise fulfilling of the law of modulation frequently proves to be impossible. Analogous difficulties are in pulse-coherent systems when according to the appropriate law must be changed the pulse amplitudes in the packet.

Somewhat more simply proceeds matter during the

permission/resolution in the time, but not in the frequency when bell-shaped form must have not envelope, but the amplitude spectrum of signal. Retaining the envelope of rectangular, it is possible, in the principle, to fulfill the necessary spectrum due to the special law of ChM within the impulse/momentum/pulse (see Chapter 8). But also in this case technical difficulties are sufficiently great.

In connection with that presented frequently is applied the further processing of signals in the receiver, which ensures the necessary permission/resolution in the frequency or in the time, but connected with some energy losses, weight processing. The operating principle of similar devices/equipment is clarified in Fig. 3.1.

From the output of UPCh the signal with rectangular envelope enters the modulator that gives to this signal the bell-shaped form  $w(t)$  (Fig. 3.1a). To carry out amplitude modulation in the receiver at the low power, obviously, it is simpler than in the transmitter. It suffices to use, for example, a temporary/time gain control according to the necessary law. The signal of the rounded off form  $w(t)$  possesses the low remainders/residues of the spectrum out of the assigned band, which allows for the frequency analyzer (set/dialing of narrow-band filters) to work with the necessary selectivity.

For increasing the selectivity in the time analogous operations are produced with the spectrum of signal (Fig. 3.1b). The rectangular spectrum is supplied to the filter with the bell-shaped characteristic  $w(\omega)$ , which gives the decrease of remainders/residues in the temporary/time representation<sup>1</sup>.

FOOTNOTE 1. For the realization of the processing indicated it is necessary to know the time of the arrival of signal (Fig. 3.1a) or its Doppler frequency (Fig. 3.1b). These conditions frequently are satisfied, since for the large duration of signal (packet), the displacement due to the unknown range is negligibly small. It is analogous for the range finders, which use signals with the wide spectrum, are low Doppler frequency shifts. ENDFOOTNOTE.

The selection of optimum weight function  $w(t)$  (or  $w(\omega)$ ) it represents fairly complicated problem. Here necessary is a compromise between the selectivity and the energy losses. Some authors assume that optimum give Dolph-Chebyshev type functions, which possess as we saw, by maximum selectivity. In particular, in the work of Temes [75] is assumed that optimum gives function of form (2.23) and are found out the adequate/approaching approximations it. However, from the following it follows that optimum weight function can noticeably

differ in its structure from the signal with the maximum selectivity. This difference is connected with the further requirement of the minimization of losses and it is substantial for the systems with the weight processing.

In this chapter is given a strict solution of the problem about optimum weight function [12].

### 3.1. Losses during weight processing.

Ideal frequency analyzer (approximation/approach to which is the set/dialing of narrow-band filters, utilized usually for the frequency selection) puts out the spectral function of the voltage, which acted to its input. Examining for the concreteness diagram in Fig. 3.1a, let us suppose that to the input of device/equipment comes the signal of constant amplitude, which has Doppler frequency  $\Omega_0$ . Taking into account amplitude modulation according to the law of  $w(t)$ , realized with the reception/procedure, spectral function will take the form

$$\tilde{w}(\omega) = \int_{-T/2}^{T/2} w(t) e^{j(\Omega_0 - \omega)t} dt.$$

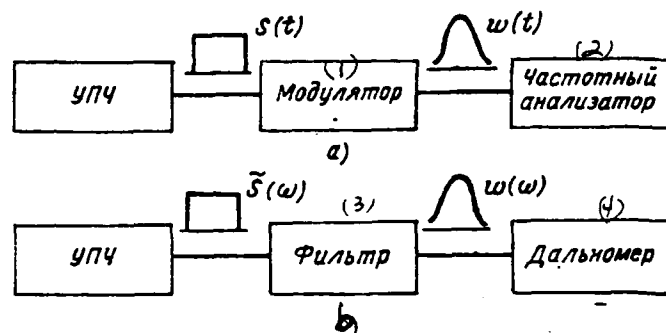


Fig. 3.1.

Key: (1). Modulator. (2). Frequency analyzer. (3). Filter. (4). Range finder.

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According to condition  $w(t)$  it is positive; therefore for the maximum overshoot of the spectrum (greatest level at the output of the corresponding narrow-band filter) we have

$$\tilde{w}_{max} = \int_{-T/2}^{T/2} w(t) dt. \quad (3.1)$$

Now let us assume that to the input of device/equipment acted the noise voltage  $n(t)$ . This voltage is also modulated of the amplitude by weight function and for the output effect we obtain respectively

$$\tilde{n}(\omega) = \int_{-T/2}^{T/2} n(t) w(t) e^{-j\omega t} dt.$$

Assuming noise  $n(t)$  white with density  $N$  W/Hz, it is not

difficult to show that the mean square of noise component at the output of analyzer is equal to

$$\overline{|\tilde{n}|^2} = N \int_{-T/2}^{T/2} w^2(t) dt. \quad (3.2)$$

The comparison of formulas (3.1) and (3.2) makes it possible to determine energy relation signal/noise at the output of the analyzer:

$$\rho = \frac{1}{N} \frac{\left[ \int_{-T/2}^{T/2} w(t) dt \right]^2}{\int_{-T/2}^{T/2} w^2(t) dt}. \quad (3.3)$$

Applying to the numerator of this formula Schwarz-Buniakowski's inequality, we find face side for value  $\rho$ :

$$\rho \leq \frac{1}{N} \frac{\int_{-T/2}^{T/2} w^2(t) dt \cdot \int_{-T/2}^{T/2} dt}{\int_{-T/2}^{T/2} w^2(t) dt} = \frac{T}{N}.$$

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From the properties of the relationship/ratio of a Schwarz-Buniakowski it follows that is here achieved the equality only by satisfaction of condition  $w(t) = \text{const}$  with  $-T/2 < t < T/2$  and, therefore, the maximum excess of the signal above the noise is obtained for rectangular  $w(t)$ . In this case, as it follows from (3.3),

$$\rho = \rho_{\text{max}} = \frac{T}{N}.$$

Thus, the energy losses, caused by nonrectangularity  $w(t)$ , are

characterized by value

$$q = \frac{P}{P_{max}} = \frac{\left[ \int_{-T/2}^{T/2} w(t) dt \right]^2}{T \int_{-T/2}^{T/2} w^2(t) dt}. \quad (3.4)$$

From previous it is clear that value  $q$  does not exceed one and is equal to it only for rectangular weight function. Analogous relationship/ratio for the case of processing in the frequency, but not on the time (diagram in Fig. 3.1b) is given by Temes [75].

### 3.2. Use/application of hypothesis of proximity.

The synthesis of optimum weight function has much in common with the problem about the signals with the maximum selectivity. As earlier, we attempt to obtain function  $w(t)$  whose spectrum is maximally concentrated in the assigned band  $(-\Omega, \Omega)$ . Best anything would be have strictly limited on spectrum band after weight processing. This means that many desired functions  $Y$  contain as in the previous problem, all functions, limited on the band, i.e.,  
 $y(\omega) = 0$  <sup>with</sup>  $|\omega| > \Omega$ .

The permissible functions have final duration, i.e., set  $X$  is characterized by the condition

$$x(t) = 0 \text{ with } |t| > T/2. \quad (3.5)$$

But we must bound also energy losses during the processing. In other words, permitted are only those weight functions, for which value  $q$ , expressed by formula (3.4), is constant:

$$q = \text{const.} \quad (3.6)$$

This further limitation differs the problem in question from the synthesis of signal with the maximum selectivity.

According to the criterion of proximity should be found the function, which belongs to set  $X$  [i.e. satisfying conditions (3.5) and (3.6)], that is located on the shortest distance

$$d_{\min} = \min_{\substack{x \in X \\ y \in Y}} d(x, y)$$

from set  $Y$ . In §2.1 it was established/installed, that in space  $L^2$  this condition satisfies the function the containing maximum part of its energy in the band  $(-\Omega, \Omega)$ . Since this conclusion of §2.1 uses only properties of set  $Y$  and it is suitable for any the  $X$  hypothesis of proximity it leads to the following variational problem.

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It is necessary to determine function  $w(t)$ , different from zero in the interval  $(-T/2, T/2)$ , which maximizes the partial energy

$$E_2 = \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} w(t) w(t') G_2(t, t') dt dt' \quad (1.7)$$

under the further conditions

$$E = \int_{-T/2}^{T/2} x^2(t) dt = 1 \quad (3.8)$$

and

$$A = \int_{-T/2}^{T/2} w(t) dt = \text{const.} \quad (3.9)$$

Kernel  $G_2(t, t')$  is determined by formula (2.6). Condition (3.8) is a usual requirement of standardization in  $L^2$ . Condition (3.9) is equivalent to (3.6), since, as from (3.4), value  $q$  is clearly uniquely determined by integral (3.9) with the fulfillment of standardization (3.8).

### 3.3. Solution of variational problem.

According to the rule of Lagrange's factors maximum to functional (3.7) under the conditions (3.8) and (3.9) gives function  $w(t)$ , which is the extremal of another functional, namely

$$\Phi = E_2 - \mu E + \nu A = \max, \quad (3.10)$$

where  $\mu$  and  $\nu$  — indefinite factors. For the determination of this extremal let us use the following method.

Spheroidal functions  $\psi_n(\xi)$  form orthonormal set, complete in interval  $(-1, 1)$ . Therefore, introducing dimensionless time  $\xi = 2\tau/T$ , it is possible to expand arbitrary weight function in the series/row of the form

$$w(\xi) = \sum_{n=0}^{\infty} a_n \psi_n(\xi); \quad -1 < \xi < 1. \quad (3.11)$$

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The parameters of problem  $E_0$ ,  $E$  and  $A$  it is not difficult to express through coefficients  $a_n$ . In this case are used the properties of spheroidal functions, examined in appendix 1. Thus, for partial energy  $E_0$  we obtain, taking into account relationships/ratios (1) and (3) of the application/appendix:

$$\begin{aligned} E_0 &= \int_{-1}^1 \int_{-1}^1 w(\xi) w(\xi') G(\xi, \xi') d\xi d\xi' = \\ &= \sum_{m,n} a_m a_n \int_{-1}^1 \int_{-1}^1 \psi_m(\xi) \psi_n(\xi') G(\xi, \xi') d\xi d\xi' = \\ &= \sum_{m,n} a_m a_n \lambda_n \int_{-1}^1 \psi_m(\xi) \psi_n(\xi) d\xi = \sum_n a_n^2 \lambda_n. \end{aligned}$$

where  $\lambda_n$  - eigenvalues.

It is analogous, total energy obtains the representation

$$E = \int_{-1}^1 w^2(\xi) d\xi = \sum_n a_n^2.$$

For converting the value  $A$  we will use relationship/ratio (8) of application/appendix. This it gives

$$A = \int_{-1}^1 w(\xi) d\xi = \sum_n a_n \int_{-1}^1 \psi_n(\xi) d\xi = \sqrt{\frac{2\pi}{c}} \sum_n a_n j_{-n} \sqrt{\lambda_n} \psi_n(0).$$

After substituting the obtained expressions in (3.10), let us register the functional being investigated in the form

$$\Phi = \sum_n \left[ (\lambda_n - \mu) a_n^2 + a_n j_{-n} \sqrt{\frac{2\pi}{c}} \lambda_n \psi_n(0) \right]. \quad (3.12)$$

We should find the values of coefficients  $a_n$  which rotate this value into the maximum. This is not difficult to do, after equating

to zero derivatives  $\partial\Phi/\partial u_n$ ; as a result it is obtained

$$a_n = \frac{\nu}{2} (-1)^n \frac{\sqrt{\frac{2\pi}{c} \lambda_n}}{\mu - \lambda_n} \psi_n(0).$$

Since spheroidal functions  $\psi_n(\xi)$  are odd with odd  $n$ , then  $\psi_{2k+1}(0) = 0$ . Therefore

$$a_{2k+1} = 0, \quad a_{2k} = \frac{\nu}{2} (-1)^k \frac{\sqrt{\frac{2\pi}{c} \lambda_{2k}}}{\mu - \lambda_{2k}} \psi_{2k}(0). \quad (3.13)$$

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Obtained coefficients  $a_n$  determine unknown optimum weight function in the form of series/rcw (3.11). We see that factor  $\nu$  enters linearly into all coefficients and affects, therefore, only the scale of the unknown function. Since the scale further is not essential, factor  $\nu$  write out we will not be.

Value  $\mu$  is determined by the permissible losses  $q$ . Actually/really, after substituting values  $a_n$  from (3.13) into the previous formulas, it is possible to obtain the following expressions for parameters  $E_2$ ,  $E$  and  $A$ , and also for extraband energy  $E' = E - E_0$ :

$$\begin{aligned}
 E_2 &= \frac{2\pi}{c} \sum_k \frac{\lambda_{2k}^2}{(\mu - \lambda_{2k})^2} \psi_{2k}^2(0), \\
 E &= \frac{2\pi}{c} \sum_k \frac{\lambda_{2k}}{(\mu - \lambda_{2k})^2} \psi_{2k}^2(0); \\
 E' &= \frac{2\pi}{c} \sum_k \frac{\lambda_{2k}(1 - \lambda_{2k})}{(\mu - \lambda_{2k})^2} \psi_{2k}^2(0); \\
 A &= \frac{2\pi}{c} \sum_k \frac{\lambda_{2k}}{\mu - \lambda_{2k}} \psi_{2k}^2(0).
 \end{aligned}
 \tag{3.14}$$

The factor of loss  $q$  is determined by relationship/ratio (3.4), which upon transfer to the dimensionless time  $\xi$  takes the form

$$q = A^2/2E. \tag{3.15}$$

As can be seen from (3.14), values  $A$  and  $E$  depend only on the parameter  $\mu$ . Therefore, fixing/recording coefficient of  $q$ , we are given implicitly the value of this parameter. On the other hand, after registering relative value of extraband energy in the form

$$p = \frac{E'}{E} = \frac{E - E_2}{E}, \tag{3.16}$$

it is possible to treat formulas (3.14) - (3.16) as the parametric form of the dependence of extraband energy (attained in the optimum case) on the losses of processing. This dependence is of great interest, since it characterizes the maximum possibilities of weight processing.

#### 3.4. Minimum losses during weight processing.

In formulas (3.14) the parameter  $\mu$  can take arbitrary values.

However, so that the coefficients  $a_n$ , appropriate (3.13), would convert the functional being investigated into the maximum, but not into the minimum, must be implemented additional condition

$$\frac{\partial^2 \Phi}{\partial a_n^2} < 0, \quad n = 0, 1, 2, \dots$$

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Differentiating (3.12), we come to relationship/ratio  $\mu > \lambda_n$ , which must be implemented for all  $n \geq 0$ . Therefore, lower boundary of the parameter  $\mu$  is determined by greatest eigenvalue  $\lambda_{\max} = \lambda_0$ . Thus, the range of values of the parameter comprises

$$\lambda_0 < \mu < \infty. \quad (3.17)$$

We trace first limiting case  $\mu \rightarrow \lambda_0$ . As it is clear from (3.13), in this case all coefficients  $a_k$  with  $k > 0$  are negligible in comparison with  $a_0$ . Therefore series (3.11) degenerates into one member, and after selecting the appropriate scale, we obtain

$$x(\xi) = \psi_0(\xi) \quad \text{with} \quad \mu \rightarrow \lambda_0. \quad (3.18)$$

Consequently, in the limiting case in question optimum weight function has the same structure as signal with the maximum selectivity.

In sums (3.14) also are retained only first terms. As a result of formula (3.15) and (3.16) they give:

$$q = \frac{\pi}{\epsilon} \lambda_0 \dot{\psi}_0^2(0); \quad p = 1 - \lambda_0. \quad (3.19)$$

Latter/last relationship/ratio, obviously, will be coordinated with the materials of the previous chapter, since for the signal with the maximum selectivity the partial energy, included in the band  $(-\Omega, \Omega)$ , is numerically equal to eigenvalue  $\lambda_0$ .

In Fig. 3.2 thin line showed the dependence of losses  $q$  on the extraband energy  $p$ , determined by formulas (3.19). This dependence is obtained as follows.

Values of  $\lambda_0$  and  $\psi_0(0)$  depend on parameter  $c = \Omega T/2$ . For eigenvalues  $\lambda_n$  are detailed tables in [68]. Furthermore, for large  $c$  is valid the asymptotic formula, used with  $n < 2c/\pi$

$$1 - \lambda_n = \frac{2^{2n+2} \sqrt{\pi} c^{n+1/2} e^{-2c}}{n!} \left[ 1 - \frac{6n^2 - 2n + 3}{32c} + O(c^{-2}) \right]. \quad (3.20)$$

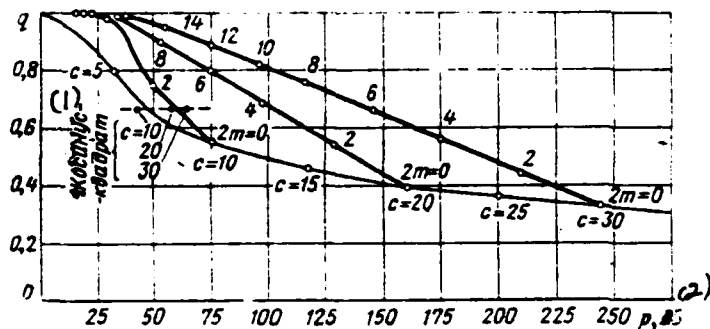


Fig. 3.2.

Key: (1). Cosine-square. (2). dB.

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Values  $\psi_n(0)$  for  $c < 5$  can be taken from tables [80] or from the graphs of Fig. 2.2. But in this case it is necessary to fulfill standardization according to condition (3) of application/appendix, (for example, by the graphical integration).

For large  $c$  we used asymptotic formula [65]:

$$\psi_n'(z) = N_n \left\{ D_n(z\sqrt{2c}) + \frac{1}{32c} [D_{n+1}(z\sqrt{2c}) - n(n-1)(n-2) \times (n-3) D_{n-1}(z\sqrt{2c})] + O(c^{-2}) \right\}. \quad (3.21)$$

This formula is applicable in region  $0 \leq z \leq \infty$ .  $D_n$  is functions of Weber (parabolic cylinder)

$$D_n(z) = 2^{-n/2} e^{-z^2/4} H_n(z/\sqrt{2}).$$

$H_n$  — Hermite's polynomial; factor  $N_n$  provides the necessary

standardization, this factor has a value

$$N_n = \frac{(c\pi)^{1/4}}{\sqrt{n!}} [1 + O(c^{-2})].$$

As a result, taking into account known relationships/ratios for Hermite's polynomials, we obtain

$$H_{2k}^2(0) = \left(\frac{c}{\pi}\right)^{1/2} \frac{(2k-1)!!}{(2k)!!} \left[1 - \frac{12k+3}{16c} + O(c^{-2})\right]. \quad (3.22)$$

Taking into account these values of formula (3.19) they take the form, convenient for the calculation:

$$q = \sqrt{\pi c} \left[1 - \frac{3}{16c} + O(c^{-2})\right].$$

$$p = 4\sqrt{\pi c} e^{-2c} \left[1 - \frac{3}{32c} + O(c^{-2})\right].$$

Let us consider the results, which escape/arise from the graph of Fig. 3.2 (thin line). If we allow losses due to the treatments, which do not exceed 1 dB ( $q \sim 0.8$ ), weight function (3.18) it provides the suppression of extraband energy to 30 dB with  $c=5$ . When such parameters are acceptable, it is inexpedient to apply another weight functions - actually, function (3.18) provides minimum extraband energy with fixed/recorded  $c$ . If however are required the smaller remainders/residues of the spectrum, then, after preserving the same structure of weight function, we are forced not only to increase the duration of signal (to increase  $c$ ), but also to allow large energy losses. For many applications/appendices this solution is undesirable, and then should be switched over to weight functions of another type, assuring smaller losses with large  $c$ . Let us limit ourselves therefore to region  $c \gg 1$ , let us switch over to research of optimum weight functions of more general/more common/more total

class, which correspond to other values of the parameter  $\mu$  from interval (3.17).

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Let us consider another limiting case  $\mu \rightarrow \infty$ . According to (3.14) and (3.15), in this case is obtained

$$E \sim \frac{2\pi}{c\mu^2} \sum_k \lambda_{2k} \psi_{2k}^2(0);$$

$$A \sim \frac{2\pi}{c\mu} \sum_k \lambda_{2k} \psi_{2k}^2(0);$$

$$q = \frac{A^2}{2E} \sim \frac{\pi}{c} \sum_k \lambda_{2k} \psi_{2k}^2(0).$$

As is known from the theory of integral equations (Mercer theorem), kernel  $G(\xi, \xi')$  allows/assumes eigenfunction expansion:

$$G(\xi, \xi') = \frac{\sin c(\xi - \xi')}{\pi(\xi - \xi')} = \sum_n \lambda_n \psi_n(\xi) \psi_n(\xi').$$

After assuming here  $\xi = \xi' = 0$ , we find

$$G(0, 0) = \frac{c}{\pi} = \sum_n \lambda_n \psi_n^2(0) = \sum_k \lambda_{2k} \psi_{2k}^2(0).$$

Therefore with  $\mu \rightarrow \infty$  it is obtained

$$q \sim \frac{\pi}{c} \cdot \frac{c}{\pi} = 1.$$

As it was established/installed in §3.1, this limiting value is achieved only by rectangular weight function. Consequently, we showed that with  $\mu \rightarrow \infty$  series/row (3.11) leads to the function of the form

$$w(\xi) = \text{const.}, \quad -1 < \xi < 1. \quad (3.23)$$

Choosing intermediate values  $\mu$ , between  $\lambda_0$  and  $\infty$ , it is possible

to fulfill transition from weight function (3.18), which has bell-shaped character, to rectangular function (3.23). In order that this transition be sufficiently smooth, we must assign to parameter  $\mu$  values very close to the eigenvalue of  $\lambda_0$ , which, in turn, insignificantly differs from unity with large  $c$ . Let us use the special rule of the selection of reading values  $\mu$ , namely let us assume

$$\mu = \mu_m = 1 + (1 - \lambda_{2m}), \quad m = 0, 1, 2, \dots \quad (3.24)$$

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This will make it possible to considerably simplify relationships/ratios (3.14), to lead them to the form, convenient for the calculations. Let us consider, for example, formula for the value  $A$ . Substituting (3.24) in (3.14), we obtain

$$A = \frac{2\pi}{c} \sum_{k=0}^{\infty} \frac{\lambda_{2k} \psi_{2k}^2(0)}{(1 - \lambda_{2m}) + (1 - \lambda_{2k})}.$$

After decomposing this sum to three parts in which  $k < m$ ,  $k = m$  and  $k > m$ , let us rewrite latter/last expression in the form

$$A = \frac{2\pi}{c(1 - \lambda_{2m})} \left[ \sum_{k=0}^{m-1} \frac{\lambda_{2k} \psi_{2k}^2(0)}{1 + \frac{1 - \lambda_{2k}}{1 - \lambda_{2m}}} + \frac{1}{2} \lambda_{2m} \psi_{2m}^2(0) + \sum_{k=m+1}^{\infty} \frac{1 - \lambda_{2m}}{1 - \lambda_{2k}} \frac{\lambda_{2k} \psi_{2k}^2(0)}{1 + \frac{1 - \lambda_{2m}}{1 - \lambda_{2k}}} \right]. \quad (3.25)$$

From asymptotic expansion (3.20) it follows that with  $k < m$

$$\frac{1 - \lambda_{2k}}{1 - \lambda_{2m}} = \frac{(2m)!}{(2k)!} (5c)^{-2(m-k)} = O[c^{-2(m-k)}].$$

... making the corrections of order  $c^{-2}$ , in the

possible to reject/throw appropriate components/terms/addends of denominators in the first sum. Analogous considerations show that should be disregarded/neglected the latter/last sum of formula (3.25). Similtude, being they are used also to other formulas (3.14), leads to the following relationships/ratios:

$$E = \frac{2\pi}{c(1-\lambda_{zm})^2} \left[ \sum_{k=0}^{m-1} \lambda_{zk} \psi_{zk}^2(0) + \frac{1}{4} \lambda_{zm} \psi_{zm}^2(0) + 0(c^{-2}) \right];$$

$$E' = \frac{2\pi}{c(1-\lambda_{zm})} \left[ \frac{1}{4} \lambda_{zm} \psi_{zm}^2(0) + 0(c^{-2}) \right]; \quad (3.25)$$

$$A = \frac{2\pi}{c(1-\lambda_{zm})} \left[ \sum_{k=0}^{m-1} \lambda_{zk} \psi_{zk}^2(0) + \frac{1}{2} \lambda_{zm} \psi_{zm}^2(0) + 0(c^{-2}) \right].$$

Calculation according to formulas (3.26) is simple. The necessary number of components/terms/addends is comparatively small. Thus, for  $c=10$  it is necessary to take not more than three members of sum ( $m \leq 2$ ), for  $c=30$  - is not more than nine ( $m \leq 3$ ). In general  $m \ll c/w$ . Values  $\lambda_{zk}$  and  $\psi_{zk}$  it is easy to count according to asymptotic formulas (3.20) and (3.22). Relationships/ratios (3.15) and (3.16) make it possible to determine further the factor of loss  $q$  and the relative level of the extraband energy  $p$ , attained at optimum weight function. For values of  $c=10, 20$  and  $30$  corresponding graphs are shown in Fig. 3.2 (heavy lines). These graphs characterize the maximum possibilities of weight treatment.

On the same figure is illustrated the case of weight treatment

according to the cosine-squared law

$$w(\xi) = \cos^2 \frac{\pi}{2} \xi; \quad -1 < \xi < 1. \quad (3.27)$$

As it is easy to check, in this case

$$E = \int_{-1}^1 w^2(\xi) d\xi = 3/4;$$

$$A = \int_{-1}^1 w(\xi) d\xi = 1;$$

$$q = \frac{A^2}{2E} = 2/3.$$

For the approximate computation of extraband energy we will use the asymptotic formula

$$\tilde{w}(\omega) = \int_{-1}^1 w(\xi) e^{-j\omega\xi} d\xi = \frac{\pi^2}{\omega^2} \sin \omega + O(\omega^{-4}).$$

escape/ensuing from the general/common/total relationships/ratios, available in [98]. Therefore:

$$E' = \frac{1}{\pi} \int_c^\infty \tilde{w}^2(\omega) d\omega \approx \frac{\pi^2}{5c^2} + O(c^{-4})$$

$$p = \frac{E'}{E} = \frac{4\pi^2}{15c^2} + O(c^{-4}).$$

FOOTNOTE 1. A precise formula for the extraband energy of the cosine-squared signal is in [23]. ENDFOOTNOTE.

For  $c=10, 20, 30$  the corresponding values  $p$  and  $q$  are shown in Fig. 3.2. It is clear that the optimization of weight functions makes it possible to very substantially decrease the remainders/residues even in comparison with this adequate/approaching law as

cosine-square.

### 3.5. Optimum weight functions.

In §3.3 were determined coefficients  $a_n$  with which series/row (3.11) it gives optimum weight function. These coefficients depend on the parameter  $\mu$  according to formula (3.13). We saw also, that, giving  $\mu$  special values  $\mu_m$  in accordance with (3.24), it is possible to considerably simplify calculations, since coefficients  $a_n$  become negligible with  $n > 2m$ .

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Using (3.13), (3.20) and (3.24), it is not difficult to show also that when  $\mu = \mu_m$  optimum weight function is expressed by the final sum

$$w(\xi) = \sum_{k=0}^{m-1} (-1)^k \sqrt{\frac{2m-k}{2m+k}} P_{m-k}(\xi) + \frac{1}{2} (-1)^m \sqrt{\frac{2m-m}{2m+m}} P_{2m}(\xi) \quad (3.25)$$

During the derivation of this formula is allowed a relative error in order  $\epsilon^{-2}$ , and scale factor  $\nu$  is selected so that would be simplified the expression. Let us note that eigenvalues  $\lambda_n$  are very close to unity.

Unfortunately, the sufficiently complete tables of spherical

functions are absent. Therefore, for some values of  $k$  and  $c$  functions  $w_m(\xi)$  in the interval  $0 \leq \xi \leq 1$  were specially designed by ETSVM [digital computer] M-20. Calculation was produced by the numerical solution of integral equation (1) of application/appendix by the method of iterations. Eigenvalues  $\lambda_m$  were taken from work [68]. As the initial approximation/approach was used the first member of asymptotic expansion (3.21).

The graphs of optimum weight functions  $w(\xi)$ , calculated by formula (3.28), they are shown in Fig. 3.3a-c. Each figure corresponds to one value of  $c$ , each curve - to one value  $m$ . The parameters of treatment  $p$  and  $q$ , attained at optimum weight functions, are depicted as the appropriate points in Fig. 3.2.

From the graphs of Fig. 3.3 it follows that optimum weight functions actually/really occupy the intermediate position between the bell-shaped signals with the maximum selectivity and the square pulse. In the middle part of function  $w(\xi)$  they are changed little, they are close to a constant value. The duration of "flat/plane part" depends on the factor of loss  $q$ . The greater  $q$ , is the more "flat/plane part" and the nearer the function  $w(\xi)$  to the rectangular. The low level of extraband energy is provided due to the exponential "fronts" of function. This gives very low values of  $w(\xi)$  near  $\xi=1$ , low jump on the edges. It is possible to show that near the

edge function  $w(\xi)$  is changed according to the law, close to the gaussian [65], and with  $\xi=1$  value  $w(\xi)$  is of the order

$$w(1) \sim c^{(m-1/2)} e^{-c}.$$

As is known, the value of jump edges substantially affects the spectrum at the high frequencies, to the extraband energy. The low value of jump, exponentially decreasing with increase of  $c$ , and makes it possible to obtain so low a level of extraband energy.

Let us pay attention also to the region of transition from the front to the flat/plane part. Here there is an oscillatory structure with the maximum overshoot of order 20% of the steady level. With increase in  $c$  the overshoot is reduced insignificantly, but oscillations/vibrations "are wrung out" to the front and occupy the low part of the complete duration. In this it is possible to perceive analogy with Gibbs's phenomenon, well known from the theory of Fourier series. Increase  $c$  is analogous, in a sense, to the expansion of band, to an increase in the number of members of Fourier series. The superposition of a large, but finite number of terms of this series/row gives, as is known, function with the the oscillatory ejections near the fronts. It is important, however, that the basis of signal near  $\xi=1$  similar oscillations/vibrations does not have any. Here the monotone decrease  $w(\xi)$  provides the low level of extraband energy.

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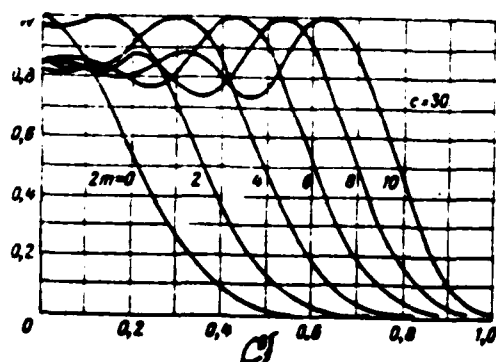
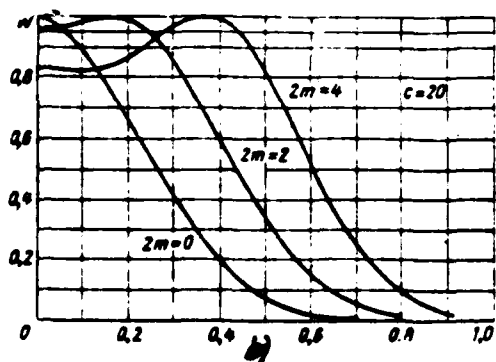
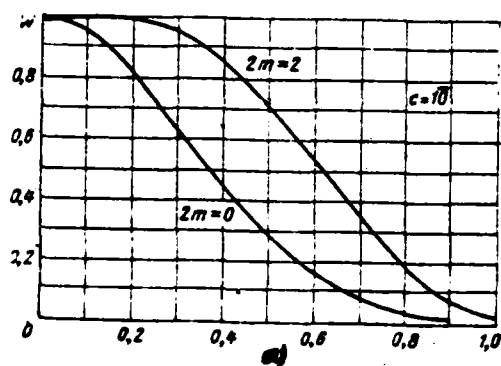


Fig. 3.3.

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Chapter 4.

#### THE SYNTHESIS OF CORRELATION FUNCTIONS.

In present chapter is examined the task of the synthesis of the realizable autocorrelation functions  $R(t)$ , which are connected with signals  $s(t)$  with the known equivalent relationships/ratios

$$R(t) = \frac{1}{E} \int_{-\infty}^{\infty} s\left(t' + \frac{t}{2}\right) s^*\left(t' - \frac{t}{2}\right) dt', \quad (4.1)$$

$$R(t) = \frac{1}{2\pi E} \int_{-\infty}^{\infty} |\tilde{s}(\omega)|^2 e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{j\omega t} d\omega, \quad (4.2)$$

where

$$g(\omega) = \frac{|\tilde{s}(\omega)|^2}{E} \geq 0 \quad (4.3)$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) d\omega = \frac{1}{2\pi E} \int_{-\infty}^{\infty} |\tilde{s}(\omega)|^2 d\omega = 1. \quad (4.4)$$

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The autocorrelation function  $R(t)$  is a response of optimum receiver of PLS (matched filter) to the signal, reflected from the fixed pinpoint target. In other words, this is an apparatus function of PLS in the case of the measurements only of range (time of arrival), it characterizes accuracy and resolution of such measurements.

We examine the task of approaching the autocorrelation function  $R(t)$  to certain desired function  $F(t)$ . Usually for the radar is substantial only the approximation/approach of the moduli/modules of the functions indicated, in connection with which, being limited to quadratic criterion, we will minimize value

$$J = \int_{-\infty}^{\infty} \{|F(t)| - |R(t)|\}^2 dt, \quad (4.5)$$

assuming/setting the spectrum of function  $R(t)$  by that limited by conditions (4.3) and (4.4). Furthermore, since usually it is possible to consider that the spectrum of signal  $\tilde{S}(\omega)$  occupies the final frequency band, let us introduce the further limitation

$$g(\omega) = 0 \quad \text{with } |\omega| > \Omega. \quad (4.6)$$

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The realizable autocorrelation function  $R(\tau)$  does not determine unambiguously signal  $s(t)$ . In accordance with (4.2) and (4.3) it is given only amplitude spectrum

$$|\tilde{S}(\omega)| = |\overline{Eg(\omega)}|,$$

while the phase spectrum of signal it remains arbitrary. Therefore realizable  $R(\tau)$  is realized by any signal with the spectrum of the form

$$\tilde{S}(\omega) = |\overline{Eg(\omega)}| e^{-j\varphi(\omega)},$$

where  $g(\omega)$  - the spectrum of the correlation function  $R(t)$ , and  $\varphi(\omega)$

- it is arbitrary. The determination of the optimum spectra  $g(\omega)$ , which satisfy criterion (4.5) under further conditions (4.3), (4.4) and (4.6), is the task of this chapter.

#### 4.1. Sets X and Y.

The permissible set X in this task, naturally, includes all realizable autocorrelation functions  $R(t)$ . This set does not fill space  $L^2$ , since in accordance with (4.3) the spectrum of the functions  $g(\omega)$  indicated is positive (it is more precise, it is non-negative), and it is also limited by conditions (4.4) and (4.6). Let us consider in somewhat more detail the property of functions  $R(t)$ .

Let signal  $s(t)$ , in general composite, be represented in the form of the sum of even and odd (composite) the component

$$\begin{aligned} s(t) &= s_1(t) + s_2(t); \\ s_1(t) &= s_1(-t); \quad s_2(t) = -s_2(-t). \end{aligned}$$

Then, as it is not difficult to show, for the energy spectrum we have

$$|\tilde{s}(\omega)|^2 = |\tilde{s}_1(\omega)|^2 + |\tilde{s}_2(\omega)|^2 + 2\operatorname{Re}(\tilde{s}_1(\omega)\tilde{s}_2^*(\omega)).$$

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Here, as it is clear from previous, first two are even functions frequencies, and the latter - odd. Thus, the in general energy

spectrum of signal has both the even and odd part. From (4.2) follows, naturally, that the autocorrelation function  $R(t)$  has real and imaginary parts, i.e.,  $R(t)$  - composite function.

However, if for all  $\omega$  is satisfied the condition

$$\operatorname{Re}(\tilde{s}_1(\omega)\tilde{s}_2^*(\omega)) \equiv 0, \quad (4.7)$$

then the spectrum of power  $|s(\omega)|^2$  is even, and  $R(t)$  - is real:

$$R(t) = \frac{1}{2\pi E} \int_{-\infty}^{\infty} |\tilde{s}(\omega)|^2 e^{j\omega t} d\omega = \frac{1}{2\pi E} \int_{-\infty}^{\infty} |\tilde{s}(\omega)|^2 \cos \omega t d\omega. \quad (4.8)$$

It is not difficult to note that this occurs for the overwhelming majority of the signals used. Latter/last condition is satisfied, at least, in the following cases:

- 1)  $s(t)$  - even function (in this case  $s_2(t) \equiv 0$ ),
- 2)  $s(t)$  - odd function (in this case  $s_1(t) \equiv 0$ ),
- 3)  $s(t)$  - the real function (in this case  $\tilde{s}_1(\omega)$  is real, and  $\tilde{s}_2(\omega)$  is imaginary);
- 4)  $s(t)$  - imaginary function (it is analogous with the previous case). These cases can be still supplemented, using the fact that  $R(t)$  is not changed during the displacement of signal on the time, and also it does not depend on initial phase. Therefore case 1) and

2) apply to signals  $s(t)$ , symmetrical of relatively arbitrary  $t_0$ , and cases 3) and 4) - to the signals, led to real changes in the initial phase.

As a result, even without submerging in the study of all conditions, under which are implemented equalities (4.7), (4.8), it is possible to take without the essential damage for the generality, that  $|\tilde{s}(\omega)|^2$  - even function, but  $R(t)$  - is real.

FOOTNOTE 1. From the commonly used signals these conditions are not satisfied by the ChM impulses/moments/pulses with symmetrical (even) law of a change in the frequency. Such signals are examined in §9.7 by another method. ENDFOOTNOTE.

In this case in accordance with (4.2) and (4.3)

$$g(\omega) = g(-\omega) \quad \text{and} \quad R(t) = R(-t). \quad (4.9)$$

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As can be seen from the following, this further limitation substantially simplifies synthesis. Summarizing the aforesaid, we will consider that  $X$  - many autocorrelation functions  $x(t) = R(t)$  whose spectra  $\tilde{X}(\omega) = g(\omega)$  are limited by conditions (4.3), (4.4), (4.6) and (4.9). For future reference it is important that determined thus  $X$  is convex. Actually/really, after assuming in accordance with

(1.38)

$$x = \tau x_1 + (1-\tau)x_2, \text{ so that} \\ g(\omega) = \tau g_1(\omega) + (1-\tau)g_2(\omega),$$

not difficult to ascertain that spectrum  $g(\omega)$  satisfies conditions (4.3), (4.4), (4.6) and (4.9), if these conditions satisfy  $g_1(\omega)$  and  $g_2(\omega)$ .

Now let us examine desired set  $Y$ . Desired modulus/module  $|F(t)|$  can be, generally speaking, selected arbitrarily. But has sense to be given only even functions

$$|F(t)| = |F(-t)|. \quad (4.10)$$

Odd component of modulus/module  $|F(t)|$ , if it is, it does not in any way affect the obtained solution.

Actually/really, designating  $\int_{-\infty}^{\infty} |F(t)|^2 dt = c$ , let us rewrite (4.5) in the form

$$f = c - 2 \int_{-\infty}^{\infty} |F(t)| |R(t)| dt + \int_{-\infty}^{\infty} |R(t)|^2 dt.$$

Varying  $R(t)$ , here it is possible to change only two latter/last components/terms/addends, moreover only one of them - integral of the form

$$\int_{-\infty}^{\infty} |F(t)| |R(t)| dt$$

- it depends on  $|F(t)|$ . The modulus/module of correlation function is even!

$$|R(t)| = |R(-t)|.$$

FOOTNOTE 1. This is correct for arbitrary  $R(t)$ , even without limitation (4.9), see (4.1). ENDFOOTNOTE.

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Therefore latter/last integral does not depend on odd component of assigned modulus/module  $|F(t)|$ . This component cannot be compensated for, selecting  $R(t)$ , it only increases general/common/total error of the approximation/approach, while the result of the solution (unknown  $R(t)$ ) it depends only on even component.

Further, according to (4.5) we attempt to obtain approximation/approach to function  $F(t)$ , assigned only on the modulus/module. The phase of this function does not play in our task of any role. In other words, all functions  $F(t)$ , that have necessary modulus/module, possess the assigned desired property. Any of such functions can be selected as the "sample/specimen" with the approximation. Therefore in accordance with the treatment of the problem of synthesis, presented in Chapter 1, we must include/connect in the desired set  $Y$  all functions of the form

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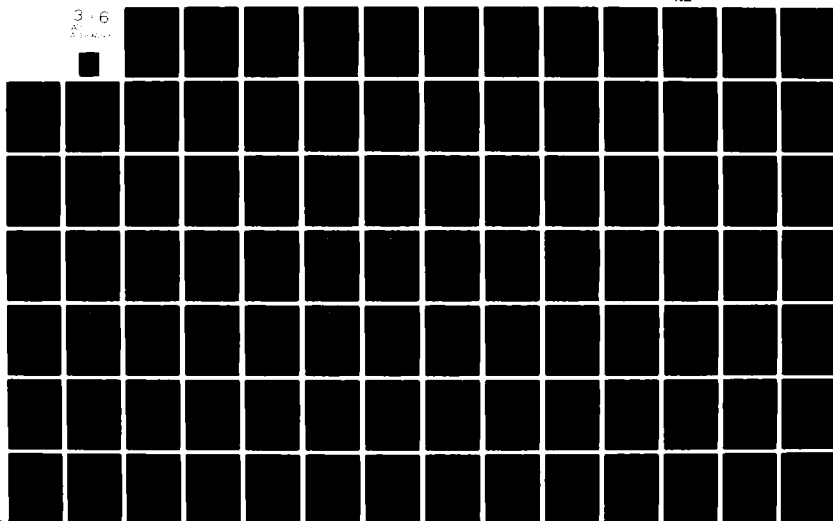
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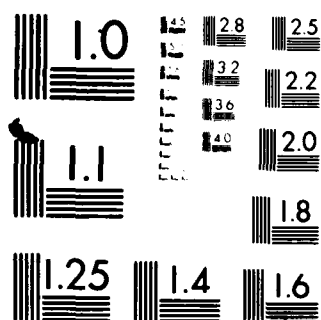
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$$y(t) = F(t) = A(t)e^{j\psi(t)}. \quad (4.11)$$

Here  $A(t) = |F(t)|$  — positive real function, which coincides with the desired modulus/module, and  $\psi(t)$  — arbitrary phase. Elements of set Y differ from each other in terms of phase functions  $\psi(t)$ .

#### 4.2. Applicability of the criterion of proximity.

Thus, being based on the essence of task, we have determined that permitted and that desired sets X and Y we can pass to the solution. According to the hypothesis of proximity the optimum correlation function  $R(t)$  is located at the shortest distance from the desired set Y, i.e., realizes the minimum of value

$$\begin{aligned} d_{\min}^2 &= \min_{\substack{R \in Y \\ F \in X}} \int_{-\infty}^{\infty} |F(t) - R(t)|^2 dt = \\ &= \min_{R \in Y} \min_{F \in X} \int_{-\infty}^{\infty} |A(t)e^{j\psi(t)} - R(t)|^2 dt. \end{aligned} \quad (4.12)$$

Let us show that the criterion of proximity (4.12) is equivalent to initial criterion (4.5), and therefore let us use to the task in question. For this we use the following order of the minimization of the distance between X and Y.

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Let us fix first arbitrary element of set Y, i.e., correlation function  $R(t)$ , and we will seek small distance of set Y, and then we will obtain  $d_{\min}$  by selecting also  $R(t)$ . The first stage corresponds

to finding value

$$\begin{aligned} d^2(x, Y) &= \min_{\psi} \int_{-\infty}^{\infty} |A(t) e^{j\psi(t)} - R(t)|^2 dt = \\ &= \min_{\psi} \left\{ \int_{-\infty}^{\infty} A^2(t) dt + \int_{-\infty}^{\infty} |R(t)|^2 dt - \right. \\ &\quad \left. - 2 \operatorname{Re} \int_{-\infty}^{\infty} A(t) R(t) e^{-j\psi(t)} dt \right\}. \end{aligned}$$

On varied phase  $\psi$  depends only latter/last component/term/addend, and we come to the maximization of value, analogous to the coefficient of the proximity

$$C = \operatorname{Re} \int_{-\infty}^{\infty} A(t) R(t) e^{-j\psi(t)} dt = \int_{-\infty}^{\infty} A(t) R(t) \cos \psi(t) dt = \max. \quad (4.13)$$

Here it is considered that in accordance with (4.9) the correlation function  $R(t)$  is real.

Further, from (4.13) we have, taking into account positiveness  $A(t)$ ,

$$C \leq \int_{-\infty}^{\infty} A(t) |R(t)| dt. \quad (4.14)$$

The right side of the latter/last inequality does not depend on varied phase  $\psi(t)$ , therefore the achievement of equality in (4.14) provides the greatest possible value  $C$ . From (4.13) and (4.14) is evident that this is achieved by satisfaction of the condition

$$\cos \psi(t) = \pm 1 = \operatorname{sign} R(t).$$

Or, which is equivalent,

$$F(t) = A(t) e^{j\psi(t)} = A(t) \operatorname{sign} R(t). \quad (4.15)$$

After determining function  $F(t)$ , which realizes the shortest distance between selected  $R(t)$  and set  $Y$ , it is possible to switch over to the second stage in which variation it undergoes by  $R(t)$ . Substituting (4.15) in (4.12), we obtain

$$\begin{aligned} d_{\min}^2 &= \min_{R \in X} \int_{-\infty}^{\infty} |A(t) \operatorname{sign} R(t) - R(t)|^2 dt = \\ &= \min_{R \in X} \int_{-\infty}^{\infty} \{ |F(t)| - |R(t)| \}^2 dt. \end{aligned}$$

Latter/last expression coincides with the initial condition for best approximation (4.5). Consequently, the criterion of proximity, being applied to the examined task, gives its full solution<sup>1</sup>.

FOOTNOTE 1. Let us note that, after foregoing condition (4.9) and assuming/setting  $R(t)$  by composite function, we would arrive at condition  $\psi(t) = \arg R(t)$  instead of (4.15). With satisfaction of this more general condition the criterion of proximity (4.12) also coincides with (4.5). Other limitations for the permissible functions  $R(t)$  in the previous conclusion/output are not used. Therefore for set  $Y$ , determined according to (4.11), the criterion of proximity is applicable with the arbitrary set  $X$ . ENDFOOTNOTE.

#### 4.3. Method of "cutting".

In the majority of locating uses/applications the desired structure of the correlation function of signal is characterized by two requirements  $R(t)$  must have sharp/acute central peak near  $t=0$  and low remainders/residues out of the assigned central region. In accordance with such requirements let us assume that assigned modulus/module  $|F(t)|$  is determined somehow in interval  $(-1,1)$ , which corresponds to central peak, and is equal to zero out of this interval.

Since interval  $(-1,1)$  must not overlap with the region of the remainders/residues (minor lobes) of correlation function, let us require so that synthesized  $R(t)$  would satisfy the condition

$$R(t) > 0 \text{ with } |t| < 1. \quad (4.16)$$

But under these assumptions formula (4.15) gives

$$F(t) = A(t) \quad (4.17)$$

and, therefore, optimum  $F(t)$  does not depend on  $R(t)$ .

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Function (4.17) belongs to the desired set  $Y$ , it is one of its elements/cells, moreover precisely this element/cell is placed at the smallest distance from arbitrary  $R(t)$ , which satisfies the stipulated

conditions, from any element/cell of the permissible set  $X$ . This is rare special feature/peculiarity - the presence of only element/cell  $y_{opt} \in Y$ , of the nearest to all those permitted  $x \in X$  - substantially simplifies further solution. As it is not difficult to comprehend, this special feature/peculiarity is caused by the structure of set  $Y$  (4.11), and also by further limitations (4.9) and (4.16), superimposed to set  $X$ .

Now, being they are confident in the fact that function  $F(t)$ , which corresponds (4.17), is located at the shortest distance from set  $X$ , we bring the task of synthesis to the simpler task of approximation. It is concrete/specific/actual, fixing/recording  $F(t)$  indicated, we they must find  $R(t)$ , which satisfies the condition

$$\begin{aligned} d^2 &= \int_{-\infty}^{\infty} |F(t) - R(t)|^2 dt = \\ &= \int_{-\infty}^{\infty} \{A(t) - R(t)\}^2 dt = \min. \end{aligned}$$

Corresponding  $R(t)$  is optimum, nearest to set  $Y$ .

After designating through  $\tilde{A}(\omega)$  the spectrum of the assigned function  $A(t)$ , on the basis of the equality of Parseval we will obtain

$$d^2 = J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\tilde{A}(\omega) - g(\omega)\}^2 d\omega = \min. \quad (4.18)$$

moreover the unknown spectrum  $g(\omega)$  must satisfy the requirements of

the feasibility of correlation function, i.e., to conditions (4.3), (4.4) and (4.6).

Most limiting from these requirements is condition (4.4), which corresponds to the standardization of energy of signal. But we will first obtain the solution without taking into account this standardization, i.e., by subordinating  $g(\omega)$  to the conditions:

$$g(\omega) \geq 0 \text{ при } |\omega| < \Omega; \quad g(\omega) = 0 \text{ при } |\omega| > \Omega.$$

Key: (1). with.

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Since during such limitations the unknown spectrum  $g(\omega)$  can take arbitrary positive values, we to the greatest degree decrease distance (4.18), if for all frequencies at which  $\tilde{A}(\omega) > 0$ , let us place  $g(\omega) = \tilde{A}(\omega)$  (in the interval  $(-\Omega, \Omega)$ ). The corresponding frequency domains in this case will not make any contribution to integral (4.18). Thus, where  $\tilde{A}(\omega) < 0$ , should be taken  $g(\omega) = 0$ . In this case the modulus/module of difference  $|\tilde{A}(\omega) - g(\omega)|$  will be minimal.

Consequently, the minimum to functional (4.18) gives following function:

$$g(\omega) = \begin{cases} \tilde{A}(\omega) & \text{при } \tilde{A}(\omega) > 0 \text{ и } |\omega| < \Omega; \\ 0 & \text{при } \tilde{A}(\omega) < 0 \text{ или } |\omega| > \Omega. \end{cases} \quad (4.19)$$

Key: (1). with. (2). and.

appropriate the "cutting" of the negative values of the assigned spectrum.

FOOTNOTE 1. In the work [7, pp 169-170] there was proposed another solution of analogous problem during the same limitations. But this solution does not give the best approximation, and, strictly speaking, it should be recognized erroneous. The verification test showed that relationship/ratio (4.19) leads to the best results.

ENDFOOTNOTE.

As a result, we come to the following procedure of synthesis (Fig. 4.1):

1. From assigned  $A(t)$  is computed spectrum  $\tilde{A}(\omega)$ .
2. Spectrum  $g(\omega)$  is formed by path "cuttings" of negative values  $\tilde{A}(\omega)$  and limitation in assigned frequency interval.
3. By inverse transformation of Fourier from  $g(\omega)$  is located unknown  $R(t)$ .

As an example let us consider synthesis  $R(t)$ , nearest to the rectangular function

$$A(t) = \begin{cases} 1 & \text{при } |t| < 1; \\ 0 & \text{при } |t| > 1. \end{cases} \quad (4.20)$$

Key: (1). with.

Spectrum of this function is of alternating sign

$$\tilde{A}(\omega) = \frac{2 \sin \omega}{\omega}.$$

Afterward the "cuttings" of the negative values of this spectrum with inverse transformation of Fourier were found the nearest correlation function. They are shown in Fig. 4.2 for the values  $\Omega=2; 4$  and  $100$ .

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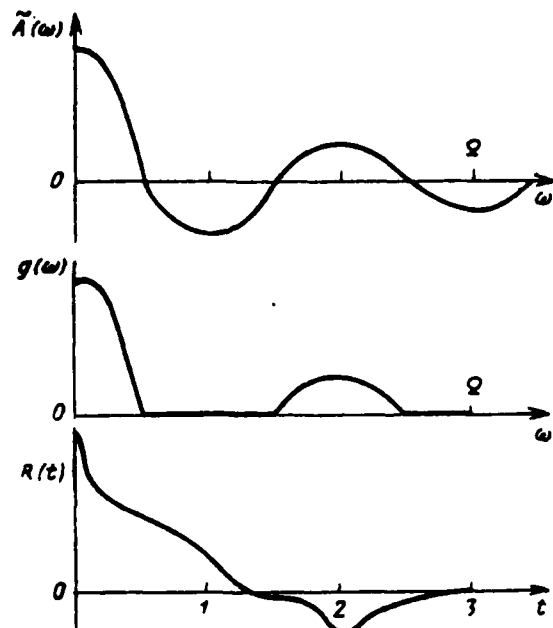


Fig. 4.1.

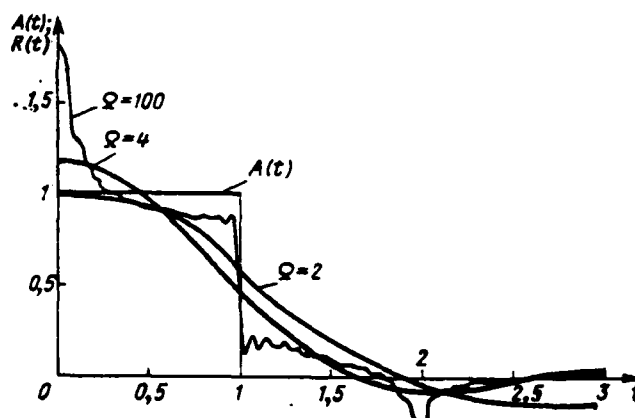


Fig. 4.2.

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Let us note that in the integral of form  $\frac{1}{2\pi} \int \tilde{A}(\omega) d\omega$  the negative values  $\tilde{A}(\omega)$  partially compensate positive ones. With formation/education  $g(\omega)$  these negative values cut themselves. Therefore beginning  $Q=2$  value

$$R(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) d\omega$$

it exceeds  $A(0)=1$ , and this excess is the greater, the more  $Q$  (with  $Q \rightarrow \infty$  value  $R(0)$  infinitely grows). As a result, the better the

approximation/approach to the assigned function, the greater the required energy of the signal, obtained with this approximation method.

#### 4.4. Account of the limitation of energy. Simplex method.

We will now minimize functional (4.18) upon consideration of all limitations to permissible  $R(t)$ , in other words, we will seek  $P(t)$ , nearest to optimum  $F(t) = A(t)$ , assuming condition (4.4) performed, i.e., normalizing energy of signal.

Let us first of all note that quadratic functional (4.18) convex, since for any  $\tau$  of the interval of  $(0, 1)$  occurs the inequality

$$\begin{aligned} \int \tau g_1(\omega) + (1-\tau)g_2(\omega) &\leq \\ &\leq \tau \int g_1(\omega) + (1-\tau) \int g_2(\omega). \end{aligned}$$

If we assume in (4.18)  $g(\omega) = \tau g_1(\omega) + (1-\tau)g_2(\omega)$ , it is possible to be convinced of the validity of this inequality. As it was noted in §4.1, the permissible set the  $X$ , which includes unknown spectrum  $g(\omega)$ , is also convex. Consequently, task consists of the minimization of convex functional on the convex set.

In order to use known iterative methods, let us reduce the task in question to the problem of square programming. For this, after decomposing the assigned frequency interval into the arbitrarily low

strips by width  $\Delta\omega$ , let us switch over to the discrete/digital representation of the spectra

$$g = (g_1, g_2, \dots, g_n),$$

where  $g_i$  — readings of function  $g(\omega)$ , undertaken with the interval  $\Delta\omega$ .

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After replacing further integrals with sums, we come to the following task:

it is necessary to find the minimum of the functional

$$f(g) = \sum_{i=1}^n (\tilde{A}_i - g_i)^2 \frac{\Delta\omega}{2\pi} \quad (4.21)$$

during the limitations

$$\sum_{i=1}^n g_i \frac{\Delta\omega}{2\pi} = 1, \quad (4.22)$$

$$g_i \geq 0; i = 1, 2, \dots, n. \quad (4.23)$$

For the solution of this problem we will use simplex method [31, 32], convenient during the linear limitations, in particular (4.22) and (4.23). The use/application of the simplex method is connected with the consecutive approximation of assigned functional (4.21) with linear. After selecting certain initial approximation/approach  $g^0 \equiv X$ , we assume/set

$$f(g) \approx f(g^{(0)}) + c_1(g_1 - g_1^{(0)}) + c_2(g_2 - g_2^{(0)}) + \dots + c_n(g_n - g_n^{(0)}) = c_0 + c_1 g_1 + c_2 g_2 + \dots + c_n g_n, \quad (4.24)$$

where  $c_i (i > 0)$  there are partial derivatives at point  $g^{(0)} \in X$ :

$$c_i = \left. \frac{\partial f(g)}{\partial g_i} \right|_{g=g^{(0)}} = (g_i^{(0)} - \tilde{A}_i) \frac{\Delta \omega}{\pi}.$$

For the determination of the point of the minimum of linear functional (4.24) during limitation (4.22) should be selected one of the variable/alternating, for example  $g_j$  as that resolving", after assuming

$$g_j = \frac{2\pi}{\Delta \omega} - \sum_{i=1}^n g_i; \quad i \neq j. \quad (4.25)$$

and to carry out minimization through the remaining  $n-1$  variable/alternating. After substituting (4.25) in (4.24), we will obtain

$$f(g) = \frac{2\pi}{\Delta \omega} c_j + \sum_{i=1}^n (c_i - c_j) g_i, \quad i \neq j. \quad (4.26)$$

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"resolving" variable/alternating  $g_j$  should be taken so that the coefficients with all unknowns would prove to be non-negative. For this, obviously, it is necessary to select the number  $j$ , which corresponds to smallest  $c_i$ , i.e.

$$c_j = \min_i \{c_i\} = \frac{\Delta \omega}{\pi} \min_i (g_i^{(0)} - \tilde{A}_i). \quad (4.27)$$

During this selection next approximation/approach will satisfy

condition (4.23) and, therefore, iterations will not deduce for the permissible set  $X$ . Actually/really, the minimum to functional (4.26) under the conditions (4.22) and (4.27), gives vector  $g$ , which has only one nonzero coordinate:

$$g = \frac{2\pi}{\Delta\omega} (0, 0, \dots, 0, 1, 0, \dots, 0), \quad (4.28)$$

moreover 1 is located on  $j$ -th position.

Vectors  $g^{(0)}$  and  $g$  are given the simplex direction, in which is realized the space of value  $\alpha$ ,

$$g^{(1)} = g^{(0)} + \alpha(g - g^{(0)}); \quad 0 \leq \alpha \leq 1. \quad (4.29)$$

As can easily be seen, with any  $\alpha$  of segment  $(0,1)$  all components of vector  $g^{(1)}$  are non-negative. The optimum length of space  $\alpha_{opt}$  is found from the condition of the minimum of initial functional (4.21) on the selected direction. After substituting (4.29) in (4.21) and differentiating on  $\alpha$ , we come to the condition

$$\alpha_{opt} = \frac{-\sum_{i=1}^n (\tilde{A}_i - g_i^{(1)}) (g_i^{(1)} - g_i)}{\sum_{i=1}^n (g_i^{(0)} - g_i)^2}$$

or, taking into account (4.28),

$$\alpha_{opt} = \frac{-\sum_{i=1}^n (\tilde{A}_i - g_i^{(0)}) g_i^{(0)} + \frac{2\pi}{\Delta\omega} (\tilde{A}_j - g_j^{(0)})}{\sum_{\substack{i=1 \\ i \neq j}}^n (g_i^{(0)})^2 + \left(g_j^{(0)} - \frac{2\pi}{\Delta\omega}\right)^2}.$$

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Further instead of  $g^{(0)}$  is taken  $g^{(1)}$  and iterations are repeated.

FOOTNOTE 1. If at certain space  $a_{ij}$  proved to be more than 1 or less than 0, should be taken 1 or 0 respectively. ENDFOOTNOTE.

Since value  $a_{opt}$  corresponds to the minimum of functional in the selected direction, the obtained sequence of approximations/approaches gives the monotone decrease of the values of functional - the distance between by X and Y. This sequence unavoidably leads to the only shortest distance, since functional (4.21) and permissible set Y are convex, and iterations are realized in the limits of this set. The major advantage of the simplex method of that consists, that the intermediate solutions at each space satisfy conditions (4.22), (4.23), i.e., belong to the permissible convex set.

Fig. 4.3 shows the correlation function, nearest to rectangular function (4.20), obtained by a simpuleks-method, and also the solution, found in the previous paragraph with the method "cuttings", that not considering the standardization of energy.

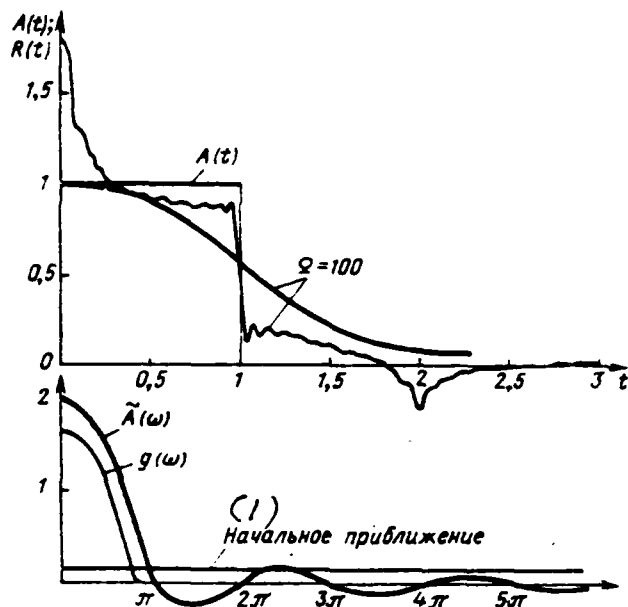


Fig. 4.3.

Key: (1). Initial approximation/approach.

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Both solutions correspond to  $Q=100$ . As is evident, the standardization of energy of signal significantly changes the structure of the solution. We obtained the bell-shaped form of function  $R(t)$ , close to ideally-spherical function  $\psi_0(t)$ , which was revealed in §2.5 as the solution of a similar problem under a somewhat distinct normalization condition<sup>1</sup>.

FOOTNOTE 1. In §2.5 optimum autocorrelation functions were considered as the generalization of signals with the maximum selectivity. For the quadratic criterion the corresponding standardization takes the form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g^2(\omega) d\omega = 1,$$

which is close to condition (4.4), but it is not identical with it.  
ENDFOOTNOTE.

Similar to spherical function, the obtained autocorrelation function  $R(t)$  has the limited on the extent spectrum (see lower graph in Fig. 4.3), although in this case were allowed/assumed the values  $\omega \leq 100$ , spectrum  $g(\omega)$  it proved to be limited in the band, which does not exceed  $\pi$ . Virtually spectrum  $g(\omega)$  completely "cuts itself", beginning from that frequency where the assigned spectrum  $\tilde{A}(\omega)$  for the first time takes zero value.

Let us note that was here used the iterative process not for minimization of the distance between two sets as in §1.8, but for approximation known to function  $y_{opt}$  on set  $X$ . At the same time, the element/call of desired set  $y_{opt}$  nearest to the permissible set  $X$ , it was possible to find out "analytically", without resorting to any iterations. In §4.6 it is shown that the task of synthesis in question admits also the comprehensive analytical solution, not

connected with iterations. The given results will be there obtained by another method.

#### 4.5. Synthesis of the multipeak correlation function.

As it was noted, for the majority of locating uses/applications are required single-peak correlation functions. Signals with the multipeak correlation functions (Fig. 4.4) are of interest in some special cases. Without stopping during the possible applications/appendices, let us consider questions of the synthesis of such signals, by assuming that assigned  $F(t)$  has several equidistant identical peaks, and, in view of condition (4.10), a number of peaks is odd (it is equal  $2N+1$ ). Being limited, as earlier, by the real correlation functions  $R(t)$ , we will assume/set  $F(t)$  of real, see (4.15).

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Let us first of all note that if  $F(t)$  is assigned completely, or the modulus/module and the sign, the task of synthesis in terms of a little differs from that examined. After computing  $\tilde{F}(\omega)$ , it is possible to use the simplex method, or the method of §4.6 in order to determine the nearest permissible spectrum  $g(\omega)$ . As earlier, matter is reduced to the minimization of functional (4.21) during

limitations (4.22)-(4.23), and the solution of this problem is singular. But for the single-peak correlation function we could previously establish that with positive  $F(t)$ , the approximation/approach will be best. Now this it is not possible to do. The peaks, shown in Fig. 4.4, can have different signs, and depending on their rotation will be obtained better or worse quality of approximation.

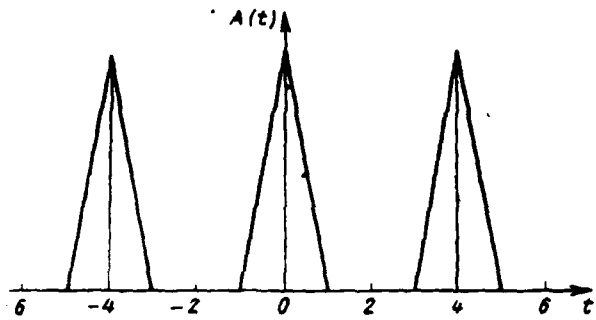


Fig. 4.4.

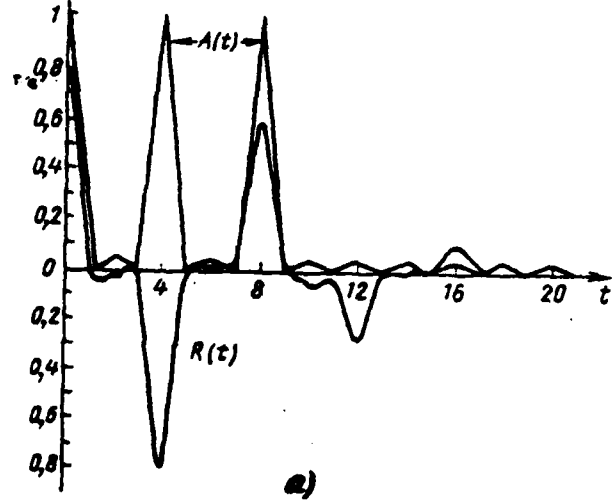


Fig. 4.5a.

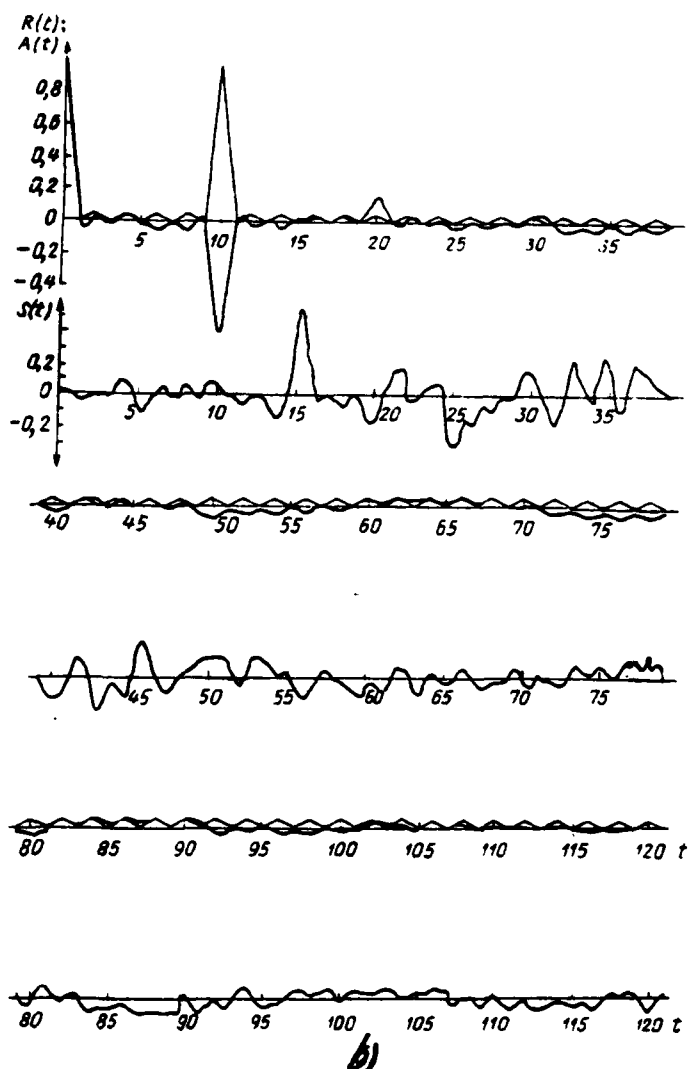


Fig. 4.5b.

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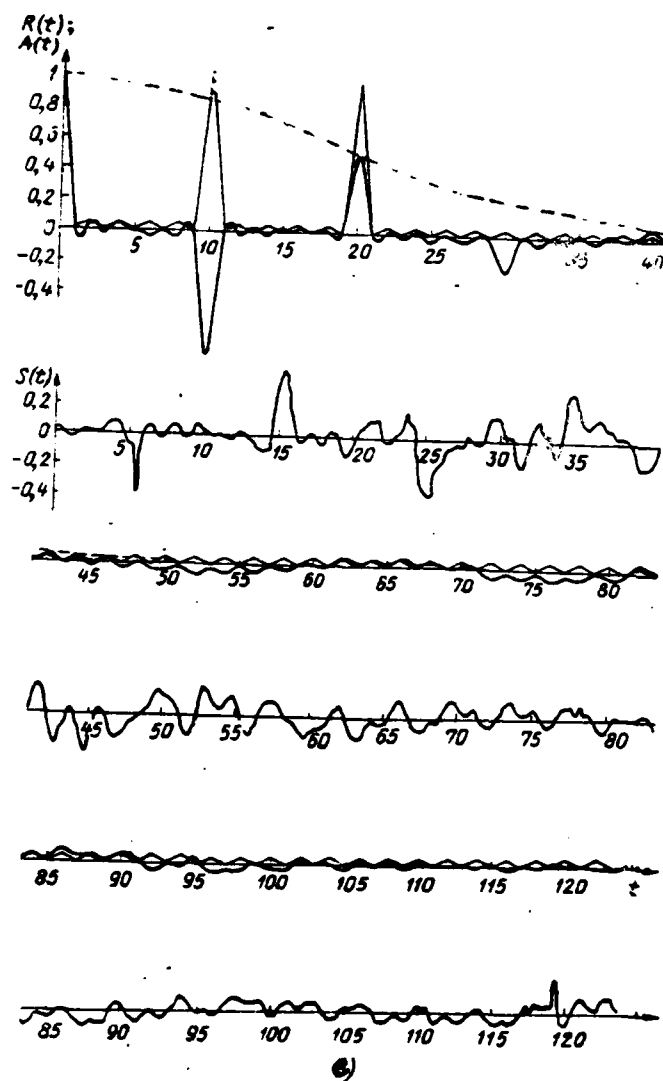


Fig. 4.5c.

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Therefore in general should be to try  $2^N$  the versions of functions  $F(t)$ , the characterized by rotation signs, and selected that, for which the error is minimum. This completely correct, the method of synthesis<sup>1</sup>.

FOOTNOTE 1. With a small number of peaks  $N$  this method is completely acceptable virtually. Moreover, since in this case is determined the spectrum of power  $\gamma(\omega)$ , but not one of the realizing signals  $s(t)$ , it should be preferred the "straight/direct" method, set forth below.

ENDFOOTNOTE.

Let us consider also another method [53]. Let us rewrite the initial criterion of approximation/approach (4.5) in the form

$$J(s) = \int \left\{ \left| F(t) \right| - \left| \int s \left( t' + \frac{t}{2} \right) s^* \left( t - \frac{t}{2} \right) dt' \right| \right\}^2 dt. \quad (4.30)$$

For the minimization of this functional according to functions  $s(t)$  it is possible to use a projective-gradient method. The limiting condition is only the standardization of the energy

$$\|s\|^2 = \int |s(t)|^2 dt = 1, \quad (4.31)$$

since any further limitations on the structure of signal we do not here set. Thus, the permissible set is the single sphere  $S$  in space  $L^2$ .

The algorithm of minimization is comparatively simple. Using general/common/total determination (1.32), it is possible to show that the gradient of functional (4.30) is equal to

$$F'(s) = \int \{|R(t')| - |F(t')|\} e^{i \arg R(t')} s(t - t') dt'. \quad (4.32)$$

Design to the single sphere  $S$  corresponds to the standardization of signal on the energy. Therefore the rule of the construction of approximations/approaches (1.34) takes the form

$$s^{(k+1)} = \frac{s^{(k)} - a_k F'(s^{(k)})}{\|s^{(k)} - a_k F'(s^{(k)})\|}. \quad (4.33)$$

Fig. 4.5 shows several signals and corresponding correlation functions, obtained by this method. The assigned modulus/modulus  $F(t)$  included three or five peaks with different distances between them. Was varied also the complete duration of the synthesized signal. For the three-peak correlation function the level of lateral peaks in all cases composes approximately/exemplarily 0.7, for the five-peak - 0.9 and 0.5 respectively. Although with this method of synthesis the quality of approximations/approaches depends on the initial signal  $s^{(0)}(t)$  (functional (4.30) has many local extrema), there are foundations for assuming that the obtained level of peaks is close to maximally possible.

#### 4.6. Use/application of Gibbs's lemma.

The previous results on synthesis of correlation functions are obtained by numerical, iterative methods, but, it proves to be, most

important of the tasks examined admit also analytical solution.

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This solution is based on Gibbs's following lemma [90]:

Let vector  $g = \{g_1, g_2, \dots, g_i, \dots\}$  minimize the function

$$f = \sum_i f_i(g_i)$$

under further conditions  $g_i \geq 0$  and  $\sum_i g_i = \text{const.}$  moreover

components/terms/addends  $f_i(g_i)$  are differentiated. Then there are a constant number  $\lambda$ , such, that

$$f'_i(g_i) = \begin{cases} \lambda_{\text{npu}} & g_i > 0 \\ 0 & g_i = 0 \end{cases} \quad (4.34)$$

Key: (1). with.

It is not difficult to note that in formulation (4.21) - (4.23) the task of synthesis in question completely corresponds to the conditions of lemma, in this case

$$f_i(g_i) = (\tilde{A}_i - g_i)^2 \frac{\Delta \omega}{2\pi}.$$

Therefore, applying (4.34), we obtain

$$f'_i(g_i) = (g_i - \tilde{A}_i) \frac{\Delta \omega}{\pi} = \begin{cases} \lambda_{\text{npu}} & g_i > 0 \\ 0 & g_i = 0 \end{cases}$$

Key: (1). with.

or, which is equivalent,

$$g_i = \begin{cases} \tilde{A}_i - \lambda_i^{(1)} & \text{при } \tilde{A}_i - \lambda_i > 0 \\ 0 & \text{при } \tilde{A}_i - \lambda_i \leq 0. \end{cases} \quad (4.35)$$

Key: (1). with.

where  $\lambda_i$  - certain new constant.

Relationship/ratio (4.23) gives the unknown solution. It shows that the optimum spectrum  $g(\omega)$ , which minimizes functional (4.21), is obtained from the assigned spectrum  $\tilde{A}(\omega)$  by certain of its displacement on the vertical line with the subsequent "cutting" of negative values (Fig. 4.6). The amount of displacement is selected so as to satisfy normalization condition (4.22). From fig 4.3 it is possible to see that the iterations according to the method of simplex directions led earlier to the same result (of course in the less common format). In order to find unknown  $R(t)$ , remains to only fulfill inverse transformation Fourier.

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As it was noted, the desired unrealizable autocorrelation function usually is assigned in the finite time interval. The alternating spectrum  $\tilde{A}(\omega)$  of this function decreases with the large ones  $\omega$ . From Fig. 4.6 it is clear that the optimum spectrum  $g(\omega)$ , which realizes best approximation, is in all such cases limited on

the band, since the high-frequency components "cut themselves". Thus, the best approximation/approach for the functions, limited in the time, proves to be limited on the frequency band. This again confirms the expedience of condition (4.6).

In this chapter we examined only the quadratic approximations/approaches of the correlation functions and their spectra. However, Gibbs's lemma leads also to the more general/more common/more total results. Actually/really, passing to the approximations/approaches in space  $L^p$ , let us consider instead of (4.21) the functional of the form

$$J = \sum_i (\tilde{A}_i - g_i)^p \frac{\Delta\omega}{2\pi} = \min.$$

Under conditions (4.22) - (4.23) to this task is also applicable Gibbs's lemma, and, as it is not difficult to show, the solution is the same as in the quadratic case. I.e., spectrum  $g(\omega)$ , constructed according to (4.35), provides best approximation in  $L^p$  with all  $p > 1$ .

This represents to very remarkable. Varying  $p$ , we included is the broad class of metrics and corresponding criteria of approximation/approach, including minimax criterion, which is obtained in the limit, with  $p \rightarrow \infty$ .

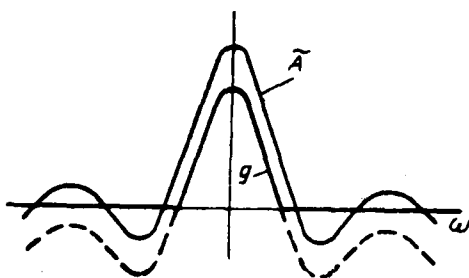


Fig. 4.6.

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With some stipulations it is possible to consider that are considered all commonly used metrics of the function spaces. In entire this class of criteria solution (4.35) proved to be universal. So complete an invariance of the solution with respect to different criteria of approximation/approach is the rare special feature/peculiarity of this task.

We spoke here about the approximation/approach of the spectra, but not quite correlation functions, that, on the whole, not one and the same. Only for the quadratic criterion, with  $p=2$ , is known the direct connection/communication between the approximations/approaches of spectra and corresponding functions of time. Specifically, this connection/communication (equality Parseval) permitted us to pass from initial criterion (4.5) for  $F(t)$  to criterion (4.18) and (4.21)

for  $g(\omega)$ .

Applying the generalization of the lemma of Gibbs (see §6.9), it is possible to show, however, that the approximation/approach in  $L^p$  correlation functions leads to the same solution (4.35). This again indicates the rare universality of the obtained solution.

## Chapter 5.

Synthesis of the functions of ~~uncertainty~~/indeterminacy.

In joint rangings and target speed of apparatus function of SLS is the function of Woodward's uncertainty/indeterminacy, represented by the following formulas:

$$\chi_s(t, \Omega) = \frac{1}{E} \int_{-\infty}^{+\infty} s\left(t' + \frac{t}{2}\right) s^*\left(t' - \frac{t}{2}\right) e^{j\Omega t'} dt'. \quad (5.1)$$

$$\chi_s(t, \Omega) = \frac{1}{2\pi E} \int_{-\infty}^{+\infty} \tilde{s}\left(\omega - \frac{\Omega}{2}\right) \tilde{s}^*\left(\omega + \frac{\Omega}{2}\right) d^{j\omega t} d\omega. \quad (5.2)$$

The task of the synthesis of signal according to the function of uncertainty/indeterminacy consists, in general terms, in the fact that is found out signal  $s(t)$ , for which  $\chi_s(t, \Omega)$  has the desired structure. This - one of the central problems of the synthesis of radar signals.

The function of uncertainty/indeterminacy  $\chi_s(t, \Omega)$  is calibrated so that

$$\chi_s(0, 0) = 1. \quad (5.3)$$

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Furthermore, occurs the invariance of the space of the body of

uncertainty/indeterminacy relative to waveform, i.e.

$$\|z_s\|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |z_s(t, \Omega)|^2 dt d\Omega = 1. \quad (5.4)$$

The latter/last property, called the uncertainty principle in the radar, substantially limits the class of the realizable functions of uncertainty/indeterminacy, so that in any way not always it is possible to find signal with that desired  $x_s(t, \Omega)$ .

Limitation (5.4) indicates, in particular, that, selecting waveform, it is not possible to ensure the arbitrarily high accuracy of joint rangings and rates.

FOOTNOTE 1. Let us emphasize that the accuracy of measurements depends also on noise level, but here we speak only about the effect of waveform. ENDFOOTNOTE.

Normalization conditions (5.3) and (5.4) are provided by factor  $1/E$  in (5.1) and (5.2). However, assuming/setting signals by those calibrated on the energy, i.e., after taking the further condition

$$E = \|s\|^2 = \int_{-\infty}^{+\infty} |s(t)|^2 dt = 1 \quad (5.5)$$

it is possible, obviously, not to write out this factor.

5.1. Geometric treatment of task.

We will examine Hilbert space  $H'$ , elements/cells of which are arbitrary functions two the variable/alternating  $F(t, \Omega)$ . In this space there is a region  $Q$ , which corresponds to many all functions of uncertainty/indeterminacy  $X(t, \Omega)$ . To this region of space  $H'$  operator (5.1) maps entire space of signals  $H$  (Fig. 5.1).

Let be assigned certain function  $F(t, \Omega)$ , which is adequate/approach from the point of view of ranging and rate. For example,  $F(t, \Omega)$  has sharp/acute central peak with  $t=\Omega=0$  and is equal to zero everywhere out of this peak. This function of uncertainty/indeterminacy is not realized. Consequently,  $F(t, \Omega)$  does not belong to  $Q$  region.

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It is possible, however, to obtain best approximation to the assigned function, if we determine projection  $F$  on  $Q$ . Thus approximation/approach to function  $F(t, \Omega)$ , assigned completely of the modulus/modulus and the phase, is reduced to the approximation in space  $H'$ . If in the space of signals  $H$  assigned certain subset of the permissible signals  $X$ , then matter is reduced to approximation  $F$  not on  $Q$ , but to subset  $X'$ , wholly included within  $Q$ . This subset

contains the functions of uncertainty/indeterminacy  $x_i(t, \Omega)$  of all permissible signals  $x(t) \in X$ . It is further necessary to find the realizing signal, i.e., to return from space  $H'$  in the initial space of signals  $H$ .

The phase of the function of uncertainty/indeterminacy does not play in the role of the significant role. Therefore designating  $|F(t, \Omega)| = A(t, \Omega)$ , it is possible to replace original function  $F(t, \Omega)$  with any function of the form

$$F(t, \Omega) = A(t, \Omega)e^{j\psi(t, \Omega)}, \quad (5.6)$$

where  $\psi(t, \Omega)$  - is arbitrary. This it means, it is obvious, that in space  $H'$  is not an only desired element/cell  $F$ , but many such elements/cells, which we will designate  $Y'$ . Functions  $F \in Y'$  have one and the same modulus/module  $A(t, \Omega)$ , but arbitrary phases  $\psi(t, \Omega)$ , and any of these functions can be selected as the "sample/specimen" with the approximation.

It is obvious, we come to the situation, characteristic for applying the hypothesis of proximity. Task is reduced to the determination of shortest distance in space  $H'$  between regions  $Y'$  and  $Q$  or, in the presence of further limitations to the permissible signals, between regions  $Y'$  and  $X'$  (moreover  $X' \subset Q$ ).

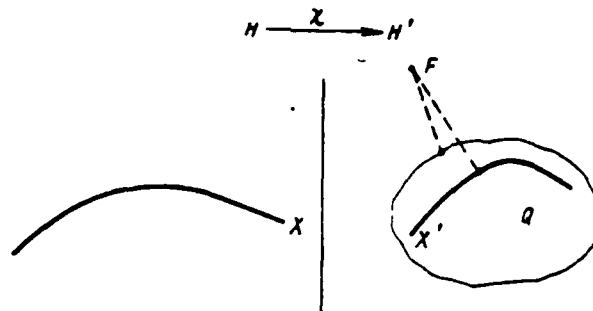


Fig. 5.1.

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This task is illustrated in Fig. 5.2 1.

FOOTNOTE 1. In contrast to Fig. 5.1 we represent now set  $Q$  in the form of one-dimensional curve on the plane. This more corresponds to the conditions of task, since set  $Q$  has less degrees of freedom than space  $H'$ . Fig. 5.1 this set depicts in the form of flat/plane region in order to simplify the image of set  $X'$ , which is part of  $Q$ . Of course all these geometric constructions are very conditional.

ENDFOOTNOTE.

Let us establish, first of all, the condition of optimum character which satisfies the function of uncertainty/indeterminacy

$\chi_{opt}(t, \Omega) \in Q$ , realizing shortest distance  $d_{min}$ .

Let in space  $H'$  be introduced the quadratic metric, i.e., the distance between functions  $F_1$  and  $F_2$  is determined in the form

$$d^2(F_1, F_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F_1(t, \Omega) - F_2(t, \Omega)|^2 dt d\Omega. \quad (5.7)$$

If are examined functions on the single sphere of space  $H'$ , for which

$$\|F\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(t, \Omega)|^2 dt d\Omega = 1, \quad (5.8)$$

then, as it is not difficult to note

$$d^2(F_1, F_2) = 2[1 - C(F_1, F_2)],$$

where

$$C(F_1, F_2) = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(t, \Omega) \overline{F_2(t, \Omega)} dt d\Omega \quad (5.9)$$

- the coefficient of proximity.

This relationship/ratio generalizes the concept of the coefficient of proximity during the function of two variable/alternating, and, as earlier, the minimization of distance is equivalent to the maximization of the coefficient of proximity.

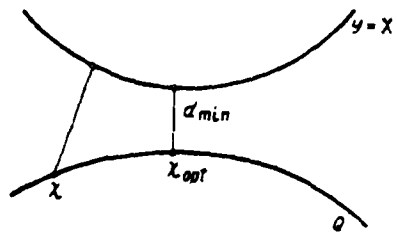


Fig. 5.2.

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Set  $Q$ , which includes the functions of the uncertainty/indeterminacy of arbitrary signals, also is placed on a unit sphere. This follows from relationship/ratio (5.4), which expresses the uncertainty principle in the radar.

Let us fix the now arbitrary function of uncertainty/indeterminacy  $x \in Q$  and we will seek shortest distance from this function to set  $Y'$ . By other words, let us determine the coefficient of the proximity

$$C(x, Y') = \max_{F \in Y'} C(x, F) = \max_{F \in Y'} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x F^* dt d\Omega.$$

Designating  $x(t, \Omega) = |x(t, \Omega)| e^{j\varphi(t, \Omega)}$  and taking into account determination (5.6), we obtain

$$\begin{aligned}
 C(\chi, Y') &= \max_{F \in Y'} \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi| \cdot |F| e^{i(\varphi - \psi)} dt d\Omega = \\
 &= \max_{\varphi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi| \cdot |F| \cos(\varphi - \psi) dt d\Omega.
 \end{aligned}$$

Of the conditions the tasks of function  $|\chi|$ ,  $|F|$  and  $\phi$  are fixed/recorded, maximization is produced according to phase functions  $\psi(t, \Omega)$ , which differ one signal of set  $Y'$  from another. But, as it follows from latter/last relationship, maximum reaches in that and only when

$$\psi(t, \Omega) = \varphi(t, \Omega). \quad (5.10)$$

We come to the following theorem:

The shortest distance between the function of uncertainty/indeterminacy  $\chi(t, \Omega)$  and set  $Y'$  realizes function  $F \in Y'$ , phase of which coincides with the phase of the function of uncertainty/indeterminacy  $\psi(t, \Omega) = \arg \chi(t, \Omega)$ .

By others by owls, design  $\chi$  to set  $Y'$  is reduced to the adding of the phase:

$$P_\gamma(\chi) = |F(t, \Omega)| e^{i \arg \gamma(t, \Omega)}.$$

If condition (5.10) is satisfied, then, as it follows (5.7), the distance between selected function  $\chi$  and set  $Y'$  comprises

$$d^2(\chi, Y') = \min_{F \in Y'} d^2(\chi, F) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|F| - |\chi|)^2 dt d\Omega.$$

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In order to obtain minimum distance  $d_{min}$  between sets  $Q$  and  $Y'$ , it is necessary, varying function  $\chi$ , to be moved on set  $Q$ . Thus, the optimum function of uncertainty/indeterminacy  $\chi_{opt}$  realizes the shortest distance

$$d_{min}^2 = \min_{\chi \in Q} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|F(t, \Omega)| - |\chi(t, \Omega)|)^2 dt d\Omega. \quad (5.11)$$

This relationship/ratio determines the condition of optimum character to which it leads the criterion of proximity.

The use/application of a criterion of proximity in space  $H'$  with metric (5.7) makes it possible to obtain best approximation on the modulus/module to assigned function in the sense of least squares criterion<sup>1</sup>.

FOOTNOTE <sup>1</sup>. From previous it is easy to note that in the proof were not used the properties of set  $Q$ . The formulated theorem is valid for any  $Q$ , not only multitude of the function of uncertainty/indeterminacy. ENDFCCTNOTE.

We used the hypothesis of proximity to a comparatively

complicated case. The minimization of distance was produced not directly in the space of signals  $H$ , but in space  $H'$ , which is connected with  $H$  by nonhomeomorphic conversion. This complication is connected with the fact that precisely in this space it is possible to determine sets  $Q$  and  $Y'$ , the distance between which characterizes the proximity of the unknown function of uncertainty/indeterminacy to the desired sample/specimen. But the hypothesis of proximity led to the completely "reasonable" criterion of approximation/approach (5.11), which corresponds to the essence of assigned mission.

5.2. Approximation/approach to the arbitrary function, assigned of the modulus/module and the phase.

We will now produce the minimization of distance in another order.

Let us fix certain function  $F(t, \Omega)$ , which belongs to set  $Y'$ , and we will seek shortest distance from this function to set  $Q$ .

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For this it is necessary to maximize the coefficient of the proximity

$$C(F, Z) = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t, \Omega) Z^*(t, \Omega) dt d\Omega \quad (5.12)$$

by selecting signal  $s(t)$ , i.e., by varying the function of uncertainty/indeterminacy  $\chi \in Q$ . Taking into account the condition for standardization (5.5), we must find the maximum of the functional

$$f(s) = C(\chi, F) - \lambda E = \max, \quad (5.13)$$

where  $\lambda$  - Lagrange's indefinite factor.

Let us compute the derivative of functional (5.12). Substituting  $s(t)$  by  $s(t) + \tau h(t)$  (where  $h(t)$  is arbitrary, and  $\tau$  - low parameter) and using determination of the function of uncertainty/indeterminacy (5.1), it is not difficult to obtain

$$\begin{aligned} \frac{\partial C}{\partial \tau} \Big|_{\tau=0} &= \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t, \Omega) e^{-j\Omega t'} \times \\ &\times s^* \left( t' + \frac{t}{2} \right) h \left( t' - \frac{t}{2} \right) dt d\Omega + \\ &+ \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t, \Omega) e^{-j\Omega t'} \times \\ &\times s^* \left( t' - \frac{t}{2} \right) h^* \left( t' + \frac{t}{2} \right) dt d\Omega. \end{aligned}$$

In the first integral let us replace the integrand of that compositely conjugated/combined. This is admissible, since further is computed the real part. Let us then replace variable/alternating integrations for the formulas

$$u = t' - \frac{t}{2}; \quad v = t' + \frac{t}{2}; \quad \Omega = -\Omega.$$

In the second integral let us replace variable/alternating according to the formulas:

$$u = t' + \frac{t}{2}; \quad v = t' - \frac{t}{2}; \quad \Omega = \Omega.$$

As a result it is obtained

$$\left. \frac{dC}{d\tau} \right|_{\tau=0} = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{+\infty} h^*(u) du \iint_{-\infty}^{+\infty} [F(u-v, \Omega) + F^*(v-u, -\Omega)] \exp\left(-j\Omega \frac{u+v}{2}\right) s(v) d\Omega dv.$$

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Somewhat more simply is computed a variation in functional (5.5)

$$\left. \frac{\partial E}{\partial \tau} \right|_{\tau=0} = 2 \operatorname{Re} \int_{-\infty}^{+\infty} h^*(u) s(u) du.$$

Therefore in accordance with (5.13)

$$\begin{aligned} \left. \frac{\partial f}{\partial \tau} \right|_{\tau=0} &= \left. \frac{\partial C}{\partial \tau} \right|_{\tau=0} - \lambda \left. \frac{\partial E}{\partial \tau} \right|_{\tau=0} = \\ &= \operatorname{Re} \int_{-\infty}^{+\infty} h^*(u) \left\{ \frac{1}{2\pi} \iint_{-\infty}^{+\infty} [F(u-v, \Omega) + F^*(v-u, -\Omega)] \times \right. \\ &\quad \times \exp\left(-j\Omega \frac{u+v}{2}\right) s(v) d\Omega dv - 2\lambda s(u) \Big\} du. \end{aligned}$$

In accordance with (1.32) the expression in the curly braces is derivative of the functional

$$\begin{aligned} f'(s) &= \frac{1}{2\pi} \iint_{-\infty}^{+\infty} [F(u-v, \Omega) + F^*(v-u, -\Omega)] \times \\ &\quad \times \exp\left(-j\Omega \frac{u+v}{2}\right) s(v) d\Omega dv - 2\lambda s(u). \end{aligned}$$

Functional  $f(s)$  reaches maximum, if  $f'(s)=0$ , i.e.

$$\int_{-\infty}^{+\infty} G(u, v) s(v) dv = \lambda s(u), \quad (5.14)$$

where kernel  $G(u, v)$  depends on the assigned function  $F(t, \Omega)$  and is determined in the form

$$G(u, v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{F(u-v, \Omega) + F^*(v-u, -\Omega)}{2} e^{-j\Omega \frac{u+v}{2}} d\Omega. \quad (5.15)$$

As it is not difficult to note that kernel hermitian:

$$G(u, v) = G^*(v, u).$$

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Thus, approximation/approach to function  $F(\tau, \Omega)$ , assigned of the modulus/module and the phase, is reduced to the solution of homogeneous equation (5.14) with hermitian kernel (5.15) <sup>1</sup>.

FOOTNOTE <sup>1</sup>. This equation is for the first time found by V. I. Dobrokhotov. ENDFOOTNOTE.

Let us explain the sense of eigenvalue  $\lambda$ . Let signal  $s(u)$  satisfy equation (5.14). Then, multiplying left and right side (5.14) to  $s^*(u)$  and integrating piecemeal, we obtain taking into account (5.5):

$$\lambda \int |s(u)|^2 du = \lambda = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G(u, v) s(v) s^*(u) du dv.$$

Or, using (5.15),

$$\begin{aligned} \lambda = & \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u-v, \Omega) s(v) s^*(u) \times \\ & \times \exp\left(-j\Omega \frac{u+v}{2}\right) du dv d\Omega + \\ & + \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F^*(v-u, -\Omega) s(v) s^*(u) \times \\ & \times \exp\left(-j\Omega \frac{u+v}{2}\right) du dv d\Omega. \end{aligned}$$

Being returned in these integrals to to the variable/alternating  $t$  and  $t'$ , i.e., implementing the replacement of variable/alternating, reverse/inverse by that used above, it is not difficult to obtain

$$\lambda = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(t, \Omega) \chi^*(t, \Omega) dt d\Omega = C(F, \chi).$$

Thus, eigenvalue  $\lambda$  is numerically equal to the coefficient of proximity. Since the task consists in the maximization of the coefficient of proximity, solution gives eigenfunction of equation (5.14), which corresponds to maximum eigenvalue  $\lambda_{\max} = \lambda_0$ . To these we demonstrated also that eigenvalues of kernel  $G(u, v)$  were limited:  $|\lambda| \leq 1$ .

From (5.1) it is clear that the function of uncertainty/indeterminacy  $\chi(t, \Omega)$  possesses the property of the symmetry

$$\chi_*(t, \Omega) = \chi^*(-t, -\Omega). \quad (5.16)$$

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Let us present assigned function  $F(t, \Omega)$  in the form of the sum

$$F(t, \Omega) = F_1(t, \Omega) - F_2(t, \Omega),$$

where

$$F_1(t, \Omega) = \frac{1}{2} [F(t, \Omega) + F^*(-t, -\Omega)] = F^*_1(-t, -\Omega),$$

$$F_2(t, \Omega) = \frac{1}{2} [F(t, \Omega) - F^*(-t, -\Omega)] = -F^*_2(-t, -\Omega).$$

If we substitute this sum in (5.12) and to take into account symmetry (5.16), it is not difficult to ascertain that the coefficient of proximity depends only on first term  $F_1(t, \Omega)$ ; value  $F_2(t, \Omega)$  does not affect value of  $C$ . Therefore assigned  $F(t, \Omega)$  expedient to subject to the condition

$$F(t, \Omega) = F^*(-t, -\Omega), \quad (5.17)$$

with which expression for kernel (5.15) is simplified

$$G(u, v) = \int_{-\infty}^{+\infty} F(u-v, \Omega) \exp\left(-j\Omega \frac{u+v}{2}\right) d\Omega. \quad (5.15a)$$

Let us consider also the degenerate case when assigned  $F(t, \Omega)$  is the realizable function of uncertainty/indeterminacy. It is possible to show that for any realizable function of uncertainty/indeterminacy is correct the identity (see §7.1).

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \gamma_s(u-v, \Omega) \exp\left(-j\Omega \frac{u+v}{2}\right) d\Omega = s(u)s^*(v). \quad (5.18)$$

Therefore, if  $F(t, \Omega)$  - the realizable function of uncertainty/indeterminacy, then kernel  $G(u, v)$  is degenerated:

$$G(u, v) = s(u)s^*(v).$$

equation (5.14) takes the form

$$\lambda s(u) = s(u) \int_{-\infty}^{\infty} |s(v)|^2 dv = s(u).$$

It is obvious, it is satisfied only with  $\lambda=1$ . Consequently, the coefficient of proximity attains one in that and only in such a case, when  $F(t, \Omega)$  - realizable function of uncertainty/indeterminacy.

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5.3. Approximation/approach to the arbitrary function, assigned on the modulus/module.

As it was shown in §5.1, approximation/approach to the function, assigned only on modulo, is equivalent to finding minimum distance  $d_{min}$  between sets  $Y'$  and  $Q$  in space  $H'$ . The criterion of approximation/approach corresponds in this case (5.11). If we fix function  $F(t, \Omega) \in Y'$ , then, by using the method of the previous paragraph, it is possible to find the distance between this function and set  $Q$ . This distance is characterized by the coefficient of proximity (5.12) and solution gives eigenfunction of integral equation (5.14), which corresponds to greatest eigenvalue  $\lambda_0$ . In order to arrive at minimum distance  $d_{min}$ , it is necessary to further lead minimization on the elements/cells of set  $Y'$ . This it means that necessary to replace function  $F(t, \Omega)$  by function  $A(t, \Omega) e^{j\psi(t, \Omega)}$  where phase  $\psi(t, \Omega)$  is arbitrary, and to select this phase in order to arrive

at the minimum of distance. Since the coefficient of proximity is equal to eigenvalue  $\lambda_0$ , we come to the following task: it is necessary to determine phase  $\psi(t, \Omega)$ , maximizing the greatest eigenvalue of kernel (5.15).

The straight/direct analytical methods of the solution of this problem are not known, and in §5.5 let us consider the appropriate iterative methods, and in §5.7 - one approximation method of the solution. But there is an important for the practice class of signals, for which the problem substantially is simplified. As showed Stutt [70], if signal  $s(t)$  is either the even or odd function of time (but not the arbitrary function, which has even and odd parts), then the function of uncertainty/indeterminacy  $\chi(t, \Omega)$  was real. Is correct reverse/inverse: if the function of uncertainty/indeterminacy is real, then signal is either the even or odd function of time.

Let it be the approximation/approach to the function, assigned on modulo, must be obtained on the subset of even and odd signals. We must ascribe to the assigned real function  $A(t, \Omega)$  phase  $\psi(t, \Omega)$  so that the approximation/approach would prove to be best.

This occurs if  $\psi(t, \Omega) = 0$ . Actually/really, let the function

$$F_{opt}(t, \Omega) = A(t, \Omega) e^{i\psi_{opt}(t, \Omega)}$$

be arranged/located on the shortest distance from set Q. According to presented in §5.1, it has the same phase, as the function of uncertainty/indeterminacy  $\chi_{opt}(t, \Omega)$ . But the latter is real as the function of the uncertainty/indeterminacy of signal from the assigned class, that also proves the expressed confirmation. Thus, in the class of even and odd signals synthesis or criterion (5.11) is reduced to the single solution of integral equation (5.14), in which

$$F(t, \Omega) = A(t, \Omega)$$

- real function.

In chapter 6 we will consider close task on this class of signals<sup>1</sup>.

FOOTNOTE 1. In chapter 4, examining the synthesis of correlation functions, we obtained analogous simplification by somewhat a broader class of signals. ENDFOOTNOTE.

#### 5.4. Method of Sussman.

Sussman proposed the method of the synthesis of signals according to the functions of uncertainty/indeterminacy, based on close prerequisites/premises [72]. It will be shown below that this

method is equivalent presented.

Let us give the first one important relationship/ratio, which also belongs to Sussman. Let there be two arbitrary signals  $s(t)$  and  $h(t)$ . We form the cross function of the uncertainty/indeterminacy these signals

$$\chi_{sh}(t, \Omega) = \frac{1}{\sqrt{E_s E_h}} \int_{-\infty}^{\infty} s(t' + t/2) h^*(t' - t/2) e^{j\Omega t'} dt'. \quad (5.19)$$

The relationship/ratio of Sussman determines two-dimensional Fourier transform from the product of two cross functions of the uncertainty/indeterminacy:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{sh}(t, \Omega) \chi_{rg}^*(t, \Omega) e^{j(\Omega t_1 - t \Omega_1)} d\Omega dt = \\ = \chi_{gh}(t_1, \Omega_1) \chi_{rs}^*(t, \Omega). \end{aligned} \quad (5.20)$$

Its proof is in [7, 72].

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Now let us pass to the presentation of the method of Sussman. Let there be function  $F(t, \Omega)$ , assigned completely on modulo and phase, and it is necessary to find signal  $S(t)$ , function of uncertainty/indeterminacy of which  $\chi_s(t, \Omega)$  gives best approximation to  $F(t, \Omega)$ , in the sense of least squares criterion. In other words,

it is necessary to minimize value

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(t, \Omega) - \chi_s(t, \Omega)|^2 dt d\Omega = \min.$$

This is equivalent to the maximization of the coefficient of proximity  $C(F, \chi)$ , expressed by formula (5.12).

Let us decompose the unknown signal  $s(t)$  along the arbitrary system of orthonormalized functions  $f_n(t)$ :

$$s(t) = \sum_{n=0}^{\infty} s_n f_n(t). \quad (5.21)$$

The function of uncertainty/indeterminacy  $\chi_s(t, \Omega)$  can be expressed through the coefficients of expansion  $s_n$ . For this let us substitute (5.21) in (5.1) and will integrate piecemeal. We will obtain

$$\chi_s(t, \Omega) = \sum_{n, m} s_n s_m^* K_{nm}(t, \Omega), \quad (5.22)$$

where

$$K_{nm}(t, \Omega) = \int_{-\infty}^{\infty} f_n(t' + t/2) f_m^*(t' - t/2) e^{j\Omega t'} dt'. \quad (5.23)$$

As it is clear from (5.23), functions  $K_{nm}(t, \Omega)$  are cross functions of the uncertainty/indeterminacy of signals  $f_n(t)$  and  $f_m(t)$ . These functions form orthonormal set in the entire plane  $(t, \Omega)$ , i.e.,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{nm}(t, \Omega) K_{ij}^*(t, \Omega) dt d\Omega = \\ = \begin{cases} 1 & \text{при } n=i, m=j; \\ 0 & \text{в остальных случаях.} \end{cases} \end{aligned} \quad (5.24)$$

Key: (1). with. (2). in remaining cases.

Therefore functions  $K_{nm}(t, \Omega)$  call derived base functions.

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For the proof of orthogonality (5.24) we will use relationship/ratio (5.20), after assuming in it  $t_1 = \Omega_1 = 0$ . We have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{nm}(t, \Omega) K_{ij}^*(t, \Omega) dt d\Omega = \\ = K_{jm}(0, 0) K_{in}^*(0, 0). \end{aligned}$$

But according to definition (5.23)

$$K_{jm}(0, 0) = \int_{-\infty}^{\infty} f_j(t') f_m^*(t') dt' = \begin{cases} 1^{(1)}_{\text{при } m=j; \\ 0_{\text{при } m \neq j.} \end{cases}$$

Key: (1). with.

where is taken into consideration the orthogonality of base functions  $f_j(t)$  and  $f_m(t)$ . Analogous relationship/ratio occurs for  $K_{in}(0, 0)$ . This proves equality (5.24).

Sussman indicated also that the system of derived base functions is complete; this makes it possible to decompose assigned function  $F(t, \Omega)$  along the system

$$F(t, \Omega) = \sum_{n, m} F_{nm} K_{nm}(t, \Omega), \quad (5.25)$$

indicated where

$$F_{nm} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t, \Omega) K_{nm}^*(t, \Omega) dt d\Omega. \quad (5.26)$$

FOOTNOTE 1. The completeness of system  $K_{nm}(t, \Omega)$  will be confirmed in

chapter 6 (see note on page 155). ENDFOOTNOTE.

Coefficients  $F_{nm}$  form the square matrix/dia, which depends only on the assigned function  $F(t, \Omega)$ . Analogous matrix/dia forms the coefficients of expansion (5.22), which are determining the function of uncertainty/indeterminacy  $\chi_s(t, \Omega)$ .

Being returned to the task of approximation/approach to the arbitrary unrealizable function  $F(t, \Omega)$ , let us substitute expansion (5.22) and (5.25) into formula (5.12), which is determining the coefficient of proximity. Taking into account orthogonality (5.24), we find

$$\begin{aligned} C(F, \chi) &= \frac{1}{2} \sum_{n, m} F_{nm} S_n^* S_m + \frac{1}{2} \sum_{n, m} F_{nm}^* S_n S_m^* = \\ &= \sum_{n, m} G_{nm} S_n S_m^* \end{aligned} \quad (5.27)$$

where  $G_{nm} = (F_{nm} + F_{mn}^*)/2$ .

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If the assigned function  $F(t, \Omega)$  possesses symmetry (5.17), then, as is not difficult to check  $G_{nm} = F_{nm} = F_{mn}^*$ .

Task is reduced, thus, to the maximization of quadratic form (5.27) under further condition  $|S_n|^2 = \sum_n S_n^2 = 1$ , expressing

standardization on energy (5.5). It is well known that this problem solves the eigenvector, which satisfies the homogeneous matrix equation

$$Gs = \lambda s \quad (5.28)$$

at maximum eigenvalue  $\lambda_{\max} = \lambda_0$ . This eigenvalue is equal to the maximum of quadratic form (5.27) i.e. to the maximum value of the coefficient of proximity and

$$C(F, \chi) = \lambda_0. \quad (5.29)$$

From that presented it is clear that integral equation (5.14) gives the solution of the same problem, that matrix equation (5.28). This task consists in the maximization of functional (5.12) under further condition (5.5) and has unique solution<sup>1</sup>.

FOOTNOTE 1. Since functional (5.12) quadratic relative to the unknown signal  $s(t)$ , see also (5.27). ENDFOOTNOTE.

Therefore equations (5.14) and (5.28) are equivalent. Using geometric analogy, it is possible to interpret the difference between the method of Sussman and our method as follows.

The introduction of orthogonal base functions  $f_n(t)$  is equivalent to the use of certain coordinate system in the space of signals  $H$ . Each signal  $s(t)$  is represented as its projections  $s_n$  on the selected axes. Simultaneously is introduced the system of orthogonal

coordinates in space  $H^1$ , matched with the reference system in space  $H$  and expressed by derived base functions  $K_{nm}(t, \Omega)$ . The assigned function  $F(t, \Omega)$  is mapped through projections  $F_{nm}$  on the axis of this system. Matrix equation (5.28) establishes the necessary conformity between coordinates  $s_n$  of the unknown vector  $s$  and coordinates  $F_{nm}$  of the assigned function, the ensuring best approximation in the sense of the selected criterion. Integral equation (5.14) gives the same conformity without the use of coordinate representation.

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Here the same difference as during recording of one and the same geometric confirmation in the vector and in the coordinate form.

The method of Sussman extends also to the approximation/approach to the function, assigned only on the modulus/module. For this is proposed the iterative process, which makes it possible to increase step by step maximum eigenvalue  $\lambda_0$  of equation (5.28). We will show further that this process is equivalent to iterations according to the method of successive design for the task, formulated in §5.3.

However, using with the coordinate representations, it is not possible to indicate explicit dependence of matrix/die  $F$  on phase

— In this respect the "vector" form of recording is preferable, which will be used in §§5.5-5.7. On the other hand, reducing of task to matrix form makes it possible to use numerical methods for solving level (5.28). Thus, two forms of the method in question mutually supplement each other.

### 5.5. Iterative methods.

Let us consider first the method of successive design in connection with the task of synthesis, formulated in §5.3, when that desired function  $F(t, \Omega)$  is assigned by its modulus/module

$$|F(t, \Omega)| = A(t, \Omega).$$

It was noted that this task was characteristic for applying the hypothesis of proximity. The desired set  $Y'$  includes functions  $F(t, \Omega)$  with the given modulus/module, and the permissible set  $X' = Q$  - all realizable functions of uncertainty/indeterminacy. The following iterative method completely corresponds to the overall diagram, presented in §1.8.

Let us select certain function of uncertainty/indeterminacy  $x_0(t, \Omega) \in Q$  and let us determine shortest distance from this function to set  $Y'$ . Let this distance be  $d_1$ . As it was shown in §5.1, function  $F_1(t, \Omega) \in Y'$ , nearest to  $x_0(t, \Omega)$ , will be formed, if we to the assigned modulus/module  $A(t, \Omega)$  ascribe the phase of the function of the

uncertainty/indeterminacy

$$F_1(t, \Omega) = P_T(\chi) = \\ = A(t, \Omega) \exp(j \arg \chi_0(t, \Omega)).$$

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Now let us fix  $F_1(t, \Omega)$  and we will seek the function of uncertainty/indeterminacy  $\chi_1(t, \Omega)$ , arranged/located on the shortest distance of  $D_2$  and  $F_1$ . In accordance with §5.2 this task is reduced to the solution of integral equation (5.14), with kernel  $G(u, v)$ , which depend on  $F_1(t, \Omega)$ . The corresponding coefficient of proximity is the greatest eigenvalue of equation  $\lambda_0$ . Equivalent result gives the solution of the matrix equation of Sussman (5.28).

Then, fixing/recording the function of uncertainty/indeterminacy  $\chi_1(t, \Omega)$ , is sought function  $F_2(t, \Omega) \in Y'$ , arranged/located on the minimum distance of  $d_3$  from  $\chi_1$ . For this to modulus/module  $A(t, \Omega)$  is assigned phase  $\arg \chi_1(t, \Omega)$ . Then, on obtained  $F_2$  is determined the function of uncertainty/indeterminacy  $\chi_2$  nearest to  $F_2$  and, etc. As usual, this process leads to descending sequence of the distances

$$d_1 > d_2 > d_3 > \dots, \quad (5.30)$$

since at each space is determined the shortest distance between certain of functions and corresponding set. This sequence is limited from below and, therefore, descends. Thus, each space of iterations gives an improvement in the approximation/approach and, after making

a sufficient number of spaces, it is possible to approach maximum distance (5.11), which characterizes best approximation to function  $F(t, Q)$ , assigned only on the modulus/module.

Usually there are several local minimums of the distance between sets  $Y'$  and  $Q$  and during the unsuccessful selection of initial approximation/approach it is possible to arrive not at the smallest of them. In that case it is possible to only begin a new series of iterations, being transmitted from another initial approximation/approach.

It is not difficult to ascertain that this iterative process completely coincides with that proposed by Sussman [72]. Thus, and this part of the method of Sussman will be coordinated with the criterion of proximity.

The resolution of integral equation (5.14) or equivalent matrix equation (5.28), connected with the determination of eigenfunctions and eigenvalues, is one of the most labor-consuming (by the space of calculations) calculating problems. Since such solution is implemented at each space of iterations, has sense to use substantially simpler gradient method for the direct maximization of the coefficient of proximity (5.12) under condition (5.5), or, which is the same thing, for the maximization of functional  $f(s)$ ,

determined by formula (5.13).

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The derivative (gradient) of the functional indicated is calculated in §5.2:

$$f'(s) = 2 \int G(u, v) s(v) dv - 2\lambda s(u),$$

where kernel  $G(u, v)$  corresponds (5.15), and eigenvalue  $\lambda$  is numerically equal to the value of the coefficient of proximity (5.12)

$$\lambda = C(s).$$

Thus,

$$f'(s) = 2 \left[ \int G(u, v) s(v) dv - C(s) s(u) \right].$$

According to the general/common/total algorithm of gradient method (1.33) the maximizing sequence takes the form

$$\begin{aligned} s^{(k+1)}(t) &= s^{(k)}(t) + \alpha_k f'(s^{(k)}) = \\ &= s^{(k)}(t) + 2\alpha_k \left[ \int G(t, v) s^{(k)}(v) dv - C(s^{(k)}) s^{(k)}(t) \right] \quad (5.31) \end{aligned}$$

It is not difficult to note that functional (5.13) being investigated quadratic relative to  $s(t)$ .

Sequence (5.31) converge to optimum signal in the version of simple iteration from any initial approximation/approach  $s^{(0)}(t)$  [33]. Simultaneously the value of the coefficient of proximity  $C$  converge to greatest its own number  $\lambda_0$ .

FOOTNOTE 1. If only  $s^{(0)}(t)$  for the orthogonally unknown signal; probability to select this signal as the initial has a measure zero.  
ENDFOOTNOTE.

As a result we come to the following procedure of "double" iterations:

1. From the initial signal  $s_c(t)$  is computed the function of uncertainty/indeterminacy  $\chi(t, \Omega)$ .

2. To assigned modulus/module  $A(t, \Omega)$  is assigned phase of this function

$$F_1(t, \Omega) = A(t, \Omega) \exp[j \arg \chi_0(t, \Omega)];$$

3. From function  $F_1(t, \Omega)$  is computed kernel  $G_1(t, \Omega)$  according to (5.15).

4. Is chosen space  $\alpha_0$  from arbitrary initial signal  $s^{(0)}(t)$  it is constructed maximizing sequence (5.31). These iterations are implemented, until the coefficient of proximity (5.12) noticeably increases.

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When the rate of increase  $C$  descends to the assigned limit,

"internal" iterations cease, the latter/last member of maximizing sequence  $s^{(k+1)}(t)$  is accepted for the signal of first approximation  $s^{(k+1)}(t) = s_1(t)$  and is produced the next (second) cycle of "external" iterations, beginning with p. 1.

It is possible to propose also the "single" iterative process, based on the gradient methods. For this we will minimize directly error (5.11), expressed through the modulus/module of the unknown function of uncertainty/indeterminacy. We have

$$\begin{aligned} \xi = d^2(\chi, Y) &= \frac{1}{2\pi} \iint [A(t, \Omega) - |\chi(t, \Omega)|]^2 dt d\Omega = \\ &= 2(1 - C) = \min. \end{aligned}$$

It is here assumed (not to the detriment of the generality) that the assigned modulus/module is subordinated to the normalization condition

$$\|A\|^2 = \frac{1}{2\pi} \iint A^2(t, \Omega) dt d\Omega = 1, \quad (5.32)$$

and the coefficient of proximity has a value

$$C = C(s) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} A(t, \Omega) |\chi(t, \Omega)| dt d\Omega. \quad (5.33)$$

This coefficient must be maximized, varying signal  $s(t)$  and satisfying further condition (5.5).

Let us compute the derivative (gradient) of functional (5.33). On signal  $s(t)$  here depends only the function of uncertainty/indeterminacy  $\chi(t, \Omega)$ . Therefore after rewriting (5.33) in

the form

$$C(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(t, \Omega) \sqrt{\chi(t, \Omega) \chi^*(t, \Omega)} dt d\Omega$$

and after replacing  $s(t)$  by  $s(t) + \tau h(t)$ , we obtain

$$\begin{aligned} \frac{dC}{d\tau} &= \frac{1}{2\pi} \iint A \frac{\chi \frac{d\chi^*}{d\tau} + \chi^* \frac{d\chi}{d\tau}}{2|\chi|} dt d\Omega = \\ &= \operatorname{Re} \frac{1}{2\pi} \iint A \frac{\chi}{|\chi|} \frac{d\chi^*}{d\tau} dt d\Omega. \end{aligned} \quad (5.34)$$

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Introducing a designation  $\arg \chi(t, \Omega) = \varphi(t, \Omega)$ , we have further

$$\begin{aligned} \frac{\chi}{|\chi|} &= e^{j \arg \chi} = e^{j \varphi(t, \Omega)}, \\ \frac{d\chi^*}{d\tau} &= \frac{d}{d\tau} \int \left\{ s^* \left( t' + \frac{t}{2} \right) + h^* \left( t' + \frac{t}{2} \right) \right\} \times \\ &\quad \times \left\{ s \left( t' - \frac{t}{2} \right) + h \left( t' - \frac{t}{2} \right) \right\} e^{-j \varphi t'} dt' = \\ &= \int s^* \left( t' + \frac{t}{2} \right) h \left( t' - \frac{t}{2} \right) e^{-j \varphi t'} dt' + \\ &\quad + \int s \left( t' - \frac{t}{2} \right) h^* \left( t' + \frac{t}{2} \right) e^{-j \varphi t'} dt' = 0. \end{aligned}$$

Let us substitute this value in (5.34) and let us assume  $\tau=0$ , then

$$\begin{aligned} \left. \frac{dC}{d\tau} \right|_{\tau=0} &= \operatorname{Re} \frac{1}{2\pi} \iint A(t, \Omega) e^{j \varphi(t, \Omega) - j \varphi t'} \times \\ &\quad \times s^* \left( t + \frac{t}{2} \right) h \left( t' - \frac{t}{2} \right) dt' dt d\Omega + \\ &\quad + \operatorname{Re} \frac{1}{2\pi} \iint A(t, \Omega) e^{j \varphi(t, \Omega) - j \varphi t'} \times \\ &\quad \times s \left( t' - \frac{t}{2} \right) h^* \left( t + \frac{t}{2} \right) dt' dt d\Omega. \end{aligned}$$

We will consider that the assigned modulus/module  $A(t, \Omega)$  possesses the symmetry

$$A(t, \Omega) = A(-t, -\Omega)$$

In §5.2 it was shown that this assumption is justified, see (5.17). In view of property (5.16) the phase of the function of uncertainty/indeterminacy  $\phi(t, \Omega)$  has a symmetry of the form

$$\phi(t, \Omega) = -\phi(-t, -\Omega).$$

Therefore if we replace integrand in the second integral with that compositely conjugated/combined, and then to change variable/alternating integrations for formulas  $t = -t_1$ ,  $\Omega = -\Omega_1$ , after simple conversions it is obtained

$$\begin{aligned} \frac{dC}{dt} \Big|_{t=0} &= 2 \operatorname{Re} \frac{1}{2\pi} \iiint A(t, \Omega) e^{i\phi(t, \Omega) - i\omega t'} \times \\ &\quad \times s^* \left( t' + \frac{t}{2} \right) h \left( t' - \frac{t}{2} \right) dt' d\Omega. \end{aligned}$$

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Finally, after doing the appropriate replacement of variable/alternating,

$$\left. \frac{dC}{ds} \right|_{s=0} = \operatorname{Re} \frac{1}{\pi} \iiint A(t' - t, \Omega) \exp \left( j\varphi(t' - t, \Omega) - j\Omega \frac{t' + t}{2} \right) s^*(t') h(t) d\Omega dt' dt.$$

In accordance with general/common/total determination (1.32) the derivative  $C'(s)$  has a value

$$C'(s) = \frac{1}{\pi} \iint A(t' - t, \Omega) \exp \left( j\varphi(t' - t, \Omega) - j\Omega \frac{t' + t}{2} \right) s(t') dt' d\Omega, \quad (5.35)$$

that it is possible to register also in the form of the scalar product

$$C'(s) = (s, z(s)), \quad (5.36)$$

where operator  $z(s)$  is determined by the relationship/ratio

$$z(s) = \frac{1}{\pi} \int A(t' - t, \Omega) \exp \left( -j\varphi(t' - t, \Omega) + j\Omega \frac{t' + t}{2} \right) d\Omega \quad (5.36a)$$

and he depends on the phase of the function of uncertainty/indeterminacy  $\varphi(t, \Omega) = \arg \chi(t, \Omega)$  of signal  $s(t)$ , and also on

the assigned modulus/module  $A(t, \Omega)$ .

Now, having a value of the derivative of the functional being investigated, it is not difficult to construct maximizing sequence. Permissible in our problem are all signals  $s(t)$  with the single energy, see (5.5), those the permissible set is the single sphere  $S$ :  $s \in S$ , if  $\|s\| = 1$ . Therefore, applying projective-gradient method (1.34), we come to the maximizing sequence

$$s^{(k+1)}(t) = P_S[s^{(k)}(t) + \alpha C'(s^{(k)})]. \quad (5.37)$$

The operator of design to single sphere  $P_S$  is reduced to the standardization of signal on the energy, (5.37) indicates the following algorithm of approximation/approach:

1. For initial signal  $s^{(0)}(t)$  is computed the function of indefiniteness  $\chi^{(0)}(t, \Omega)$  and its phase  $\varphi^{(0)}(t, \Omega)$ .

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2. In accordance with (5.35), (5.36) is computed gradient  $C'(s^{(0)})$ .

3. For selected space  $\alpha$  is computed corrected signal

$$s(t) = s^{(0)} - \alpha C'(s^{(0)}).$$

4. This signal is normalized on energy, after which it is

accepted for signal of following (first) approximation/approach  $s^{(1)}(t)$ . Then it is implemented next (second iterative loop, for which the process is repeated, beginning with p. 1, but already with signal  $s^{(1)}(t)$ , but not  $s^{(0)}(t)$ ).

By the space of calculations the latter/last method, which switches on one, but not two iterative loops, apparently, it is more economical than previous. However, both methods are comparatively unwieldy, and, as they was noted they bring, in general, to the local, but not to the global minimum of distance. Necessarily good initial approximation so that these methods of synthesis would be efficient.

5.6. Evaluation/estimate of greatest eigenvalue through the traces of kernel.

As it was shown, synthesis according to function  $F(t, \Omega)$ , given completely, on the modulus/module and the phase, consists of finding of eigenfunction  $s_0(t)$  homogeneous equation (5.14), that corresponds to greatest eigenvalue  $\lambda_0$ . Value of  $\lambda_0$  is equal to the coefficient of proximity and characterizes, therefore, the quality of the obtained approximation/approach. In this connection it is interesting to consider even before finding of eigenfunctions, what degree of approximation can be obtained for assigned  $F(t, \Omega)$ . This is useful,

in the first place, in order to rationally assign  $F(t, \Omega)$ , beginning the synthesis, and, in the second place, in order to control the results of synthesis, obtained, for example, by iterative methods. Furthermore, the results of this and following paragraphs have the general/common/total value for the synthesis of signals according to the functions of uncertainty/indeterminacy.

The necessary evaluation/estimate of the quality of approximation/approach is reduced to the evaluation/estimate of the greatest eigenvalue of  $\lambda_0$  according to the kernel of equation (5.14) and can be carried out on the base of the known positions of the theory of integral equations [46].

The iterated kernels of equation (5.14) are formed according to the recurrent rule

$$G_m(u, v) = \int G(u, \xi) G_{m-1}(\xi, v) d\xi = \int G(u, \xi) G_{m-1}^*(v, \xi) d\xi. \quad (5.38)$$

moreover  $G_1(u, v) = G(u, v)$ . The m-trace of kernel  $G(u, v)$  is the integral

$$\rho_m = \int G_m(u, u) du. \quad (5.39)$$

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Traces are connected with its eigenvalues:

$$\rho_m = \sum_{i=0}^{\infty} \lambda_i^m = \lambda_0^m + \lambda_1^m + \lambda_2^m + \dots \quad (5.40)$$

The greater the order of trace  $m$ , the less contribution introduce into value  $\rho_m$  all components/terms/addends of series/row (5.40), except the first, the greatest. Therefore as the approximate estimate for greatest eigenvalue of  $\lambda_0$  it is possible to take value

$$\lambda_0 \approx \{\rho_m\}^{1/m}. \quad (5.41)$$

Approximation/approach here is obtained with the excess and the more precisely, the greater the order  $m$ . Furthermore, the evaluation/estimate the more precise, the more rapidly decrease the eigenvalues  $\lambda_i$ .

Let us refine an error in approximation formula (5.41):

$$\delta\lambda = \{\rho_m\}^{1/m} - \lambda_0 = \left\{ \lambda_0^m + \sum_{i=1}^{\infty} \lambda_i^m \right\}^{1/m} - \lambda_0. \quad (5.42)$$

Us it will further interest the case when kernel  $G(u, v)$  is calibrated, i.e.,

$$\|G\|^2 = \int \int |G(u, v)|^2 du dv = 1. \quad (5.43)$$

Assuming/setting  $m=2$ , from (5.38) and (5.39) we have

$$\begin{aligned} \rho_2 &= \int G_2(u, u) du = \int \int G(u, \xi) G^*(u, \xi) d\xi du = \\ &= \int \int |G(u, v)|^2 dudv = 1. \end{aligned} \quad (5.43a)$$

It is clear that evaluation/estimate through the trace  $\rho_2$  gives only trivial results  $\lambda_0 \leq 1$ . Therefore let us construct higher approximation/approach, using a trace of the fourth order  $\rho_4$ . Respectively, in formula (5.42) we should consider the value of the

sum

$$\sum_{i=1}^8 \lambda_i^4$$

As a rule, for sufficient "good" kernels eigenvalues decrease very rapidly. Therefore, providing certain "reserve strength", we will assume that  $\lambda_i$  decrease with the speed of geometric progression

$\lambda_{i+1} = q\lambda_i$ ,  $q \leq 1$ . Value  $q$  can be counted, after assuming in (5.40)  $m=2$  and using (5.43a):

$$r_2 = \lambda_0^2 \sum_{i=0}^{\infty} q^{2i} = \lambda_0^2 \frac{1}{1-q^2} = 1.$$

Therefore  $q^2 = 1 - \lambda_0^2$ .

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Further we find

$$\sum_{i=1}^{\infty} \lambda_i^4 = \lambda_0^4 q^4 (1 + q^4 + q^8 + \dots) = \frac{\lambda_0^4 q^4}{1 - q^4} = \frac{\lambda_0^4 (1 - \lambda_0^2)^2}{2 - \lambda_0^2}$$

and formula (5.42) gives with  $m=4$ :

$$\delta\lambda = \frac{\sqrt{\lambda_0}}{\sqrt{2 - \lambda_0^2}} - \lambda_0.$$

The value of error  $\delta\lambda$  as function of  $\lambda_0$  is represented in Fig. 5.3. It is obvious, the calculation of greatest eigenvalue of  $\lambda_0$  according to approximation formula (5.41) is admissible with  $m=4$  in the most important range of values  $0.8 \leq \lambda_0 \leq 1$ . Specifically, in this region is provided the high accuracy of synthesis, and error in approximation formula (5.41) does not exceed 5%.

Let us move on to calculation of the spurs of equation (5.14) in which regard assuming symmetry (5.17), we will use the

simplified representation of kernel (5.15a) :

$$G(u, v) = \frac{1}{2\pi} \int F(t, \Omega) e^{-j\Omega t'} d\Omega \quad (5.44)$$

where

$$t = u - v, \quad t' = \frac{u + v}{2}.$$

In integral (5.43), which is determining the norm of kernel  $\|G\|$  it is possible to pass to the variable/alternating  $t$  and  $t'$  instead of  $u$  and  $v$  (jacobian of this conversion is equal to unity). This it gives

$$\begin{aligned} \|G\|^2 &= \iint G G^* dt dt' = \\ &= \frac{1}{(2\pi)^2} \iiint F(t, \Omega) F^*(t, \Omega') e^{j(t' - t)(\Omega' - \Omega)} d\Omega d\Omega' dt = \\ &= \frac{1}{2\pi} \iiint F(t, \Omega) F^*(t, \Omega') \delta(\Omega' - \Omega) d\Omega' d\Omega dt = \\ &= \frac{1}{2\pi} \iint |F(t, \Omega)|^2 dt d\Omega. \end{aligned}$$

Condition (5.32) is shown further, then the standardization of kernel (5.43) actually/really is implemented.

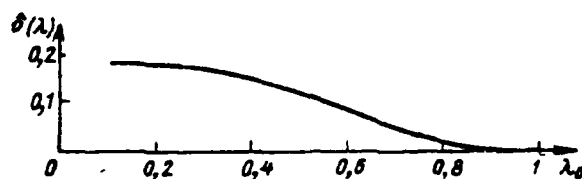


Fig. 5.3.

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The second iterated kernel  $G_2(u, v)$  we will find from (5.38) and (5.15a):

$$\begin{aligned} G_2(u, v) &= \int G(u, \xi) G^*(v, \xi) d\xi = \\ &= \frac{1}{(2\pi)^2} \iint F(u - \xi, \Omega_1) F^*(v - \xi, \Omega_2) \times \\ &\times \exp\left(-j\Omega_1 \frac{u + \xi}{2} + j\Omega_2 \frac{v + \xi}{2}\right) d\Omega_1 d\Omega_2 d\xi. \end{aligned}$$

Taking into account symmetry (5.17), this expression can be converted to the form

$$G_2(u, v) = \frac{1}{2\pi} \int L(u - v, \Omega) \exp\left(-j\Omega \frac{u + v}{2}\right) d\Omega, \quad (5.45)$$

where

$$\begin{aligned} L(t, \Omega) &= \frac{1}{2\pi} \iint F\left(t' + \frac{t}{2}, \Omega' + \frac{\Omega}{2}\right) \times \\ &\times F^*\left(t' - \frac{t}{2}, \Omega' - \frac{\Omega}{2}\right) \exp\left(j \frac{\Omega' t - \Omega t'}{2}\right) dt' d\Omega'. \end{aligned} \quad (5.46)$$

Function  $L(t, \Omega)$  plays subsequently noticeable role. In particular,

it makes it possible to compute the fourth trace of kernel  $\rho_4$ .  
Actually/really, from (5.38) and (5.39) it is possible to obtain

$$\rho_4 = \iint |G_2(u, v)|^2 du dv.$$

Therefore

$$\begin{aligned} \rho_4 &= \frac{1}{(2\pi)^2} \iiint L(u-v, \Omega_1) L^*(u-v, \Omega_2) \times \\ &\quad \times \exp\left(j(\Omega_2 - \Omega_1) \frac{u+v}{2}\right) d\Omega_1 d\Omega_2 du dv = \\ &= \frac{1}{(2\pi)^2} \iiint L(t, \Omega_1) L^*(t, \Omega_2) \exp\left(j \frac{t}{2} (\Omega_2 - \Omega_1)\right) \times \\ &\quad \times \exp(jv(\Omega_2 - \Omega_1)) dv d\Omega_1 d\Omega_2 dt = \frac{1}{2\pi} \iint |L(t, \Omega)|^2 dt d\Omega. \quad (5.47) \end{aligned}$$

Relationships/ratios (5.46), (5.47), (5.41) make it possible, in the principle, to obtain the necessary evaluation/estimate of greatest eigenvalue through the fourth trace of kernel.

#### 5.7. Equation of optimum phase.

When is assigned only modulus/module  $A(t, \Omega)$ , synthesis is produced according to the function

$$F(t, \Omega) = A(t, \Omega) e^{j\psi(t, \Omega)},$$

where phase  $\psi(t, \Omega)$  is arbitrary. Only iterative methods permitted for us to thus far find the "adequate/approaching" phase for which the approximation/approach is good.

However, the revealed analytical connection/communication of greatest eigenvalue of  $\lambda_0$  (coefficient of proximity) for assigned function  $F(t, \Omega)$  permits to additionally trace this question. The fourth trace of kernel  $\rho_*$ , determined according to (5.47), depends on phase  $\psi(t, \Omega)$ . It determines, in turn, the greatest eigenvalue of  $\lambda_0$  (however, approximately). Therefore for optimization of phase it is necessary to find maximum  $\rho_*$  from functions  $\psi(t, \Omega)$ . Substituting for this  $\psi(t, \Omega)$  on  $\psi(t, \Omega) = \gamma h(t, \Omega)$  let us compute the derivative  $d\rho_*/d\gamma$ . From (5.46) and (5.47) we obtain

$$\begin{aligned} \frac{d\rho_*}{d\gamma} &= \frac{1}{(2\pi)^2} \iint \left( L \frac{dL^*}{dt} + L^* \frac{dL}{d\tau} \right) dt d\Omega = \\ &= 2\text{Re} \iint L(t, \Omega) \frac{dL^*}{d\tau} dt d\Omega = \\ &= 2\text{Re} \frac{1}{(2\pi)^2} \iiint L(t, \Omega) F^* \left( t' + \frac{t}{2}, \Omega' + \frac{\Omega}{2} \right) \times \\ &\quad \times F \left( t' - \frac{t}{2}, \Omega' - \frac{\Omega}{2} \right) e^{-j \left( \frac{\Omega' t}{2} - \frac{\Omega t'}{2} \right)} \times \\ &\quad \times \left[ h \left( t' - \frac{t}{2}, \Omega' - \frac{\Omega}{2} \right) - h \left( t' + \frac{t}{2}, \Omega' + \frac{\Omega}{2} \right) \right] d\Omega' dt' d\Omega dt. \end{aligned}$$

Or, after the simple conversions, which consider that in accordance with (5.46)  $L(t, \Omega) = L^*(-t, -\Omega)$ ,

$$\begin{aligned} \frac{d\rho_*}{d\gamma} &= \text{Re} \frac{1}{\pi^2} \iint \text{Im} \left\{ F^*(t_1, \Omega_1) \int \int L(t_1 - t_2, \Omega_1 - \Omega_2) \times \right. \\ &\quad \times F(t_2, \Omega_2) \exp \left[ j \left( \frac{\Omega_1 t_2}{2} - \frac{\Omega_2 t_1}{2} \right) \right] dt_2 d\Omega_2 \left. \right\} h(t_1, \Omega_1) dt_1 d\Omega_1. \end{aligned} \quad (5.48)$$

It is clear that the trace  $\rho_*$  reaches maximum only with satisfaction of the condition

$$\operatorname{Im} \left\{ F^*(t_1, \Omega_1) \iint L(t_1 - t_2, \Omega_1 - \Omega_2) F(t_2, \Omega_2) \exp \left[ j \left( \frac{\Omega_1 t_2}{2} - \frac{\Omega_2 t_1}{2} \right) \right] dt_2 d\Omega_2 \right\} = 0,$$

which, as it is not difficult to show, equivalent to the following equation

$$\begin{aligned} \operatorname{tg} \psi(t_1, \Omega_1) &= \frac{\operatorname{Im} \iint L(t_1 - t_2, \Omega_1 - \Omega_2) F(t_2, \Omega_2)}{\operatorname{Re} \iint L(t_1 - t_2, \Omega_1 - \Omega_2) F(t_2, \Omega_2)} \rightarrow \\ &\rightarrow \frac{\exp \left[ j \left( \frac{\Omega_1 t_2}{2} - \frac{\Omega_2 t_1}{2} \right) \right] dt_2 d\Omega_2}{\exp \left[ j \left( \frac{\Omega_1 t_2}{2} - \frac{\Omega_2 t_1}{2} \right) \right] dt_2 d\Omega_2}. \end{aligned} \quad (5.49)$$

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This equation (containing unknown phase  $\psi$  both to the left and to the right) is determined, in the principle, optimum phase  $\psi(t, \Omega)$ , which gives maximum to greatest eigenvalue of  $\lambda_0$  with the assigned modulus/module  $A(t, \Omega)$  <sup>1</sup>.

FOOTNOTE <sup>1</sup>. Is strict this correctly for the trace  $\rho_*$ , which is only approximately connected with eigenvalue of  $\lambda_0$ . ENDFOOTNOTE.

Apparently, a similar equation is obtained for the first time, but the efficient methods of its solution proposed could not be.

# 5.8. Necessary and sufficient condition of realizability of the function of uncertainty/indeterminacy.

Known several conditions of the feasibility of the function of uncertainty/indeterminacy [7, 72] one of which will be by us obtained in §7.1. For these conditions it is characteristic that testing the realizability of the assigned function  $F(t, \Omega)$  as the functions of uncertainty/indeterminacy is reduced to finding of the realizing signal  $s(t)$ . However, using the previous results, it is possible to formulate the condition of "locked" type feasibility, which does not require the determination of the realizing signal, but which sets only certain invariance, characteristic realizable to functions.

As it will be shown, for the feasibility of function  $F(t, \Omega)$  of the function of uncertainty/indeterminacy, is necessary and sufficient the satisfaction of the condition

$$F(t, \Omega) = \frac{1}{2\pi} \iint F\left(t' + \frac{t}{2}, \Omega' + \frac{\Omega}{2}\right) \times \\ \times F^*\left(t' - \frac{t}{2}, \Omega' - \frac{\Omega}{2}\right) \exp\left[-j\left(\frac{\Omega t'}{2} - \frac{\Omega' t}{2}\right)\right] dt' d\Omega', \quad (5.50)$$

which, taking into account (5.46), can be registered also in the form

$$F(t, \Omega) = L(t, \Omega). \quad (5.50a)$$

The necessity of condition (5.50) follows from the known property of the cross functions of uncertainty/indeterminacy,

established/installed by Titlebaum [76]:

$$\frac{1}{2\pi} \iint \chi_{sh}(t' + t_1, \Omega' + \Omega_1) \chi_{sq}^*(t' - t_1, \Omega' - \Omega_1) e^{j(\Omega' - \Omega_1)t_2} dt' d\Omega' = \\ = \chi_{sr}(t_2 + t_1, \Omega_2 + \Omega_1) \chi_{sq}^*(t_2 - t_1, \Omega_2 - \Omega_1).$$

Assuming/setting all signals to identical  $s(t) = h(t) = r(t) = q(t)$ , and also  $t_1 = t_2 = t/2$ ,  $\Omega_1 = \Omega_2 = \Omega/2$ , we obtain, that the realizable function of uncertainty/indeterminacy  $\chi(t, \Omega)$  satisfies condition (5.50).

Sufficiency ensues from the following. Let  $F(t, \Omega)$  satisfy condition (5.50). We will seek signal  $s(t)$ , the function of uncertainty/indeterminacy of which  $\chi(t, \Omega)$  realizes a best approximation to  $F(t, \Omega)$ . According to presented, this signal is the solution of equation (5.14) with kernel  $G(u, v)$ , connected with  $F(t, \Omega)$  by condition (5.15a).

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The second iterated kernel of this equation  $G_2(u, v)$  is expressed as  $L(t, \Omega)$  according to (5.45). From (5.15a) and (5.45) it follows that with satisfaction of condition (5.50a) kernel  $G(u, v)$  coincides with  $G_2(u, v)$ :

$$G(u, v) = G_2(u, v). \quad (5.51)$$

From the theory of integral equations it is known that the iterated kernel of equation allows/assumes the following eigenfunction expansion:

$$G_m(u, v) = \sum_{i=0}^{\infty} \lambda_i^m s_i(u) s_i(v).$$

moreover  $G(u, v) = G_1(u, v)$ . Therefore taking into account to the orthogonality of eigenfunctions equality (5.51) can be fulfilled, only if

$$\lambda_i = \lambda_i^2; i = 0, 1, 2, \dots$$

but this is possible only for  $\lambda_i = 1$  or  $\lambda_i = 0$ . Thus, from condition (5.50) it follows that the greatest eigenvalue of equation (5.14)

$$\lambda_0 = 1. \quad (5.52)$$

This eigenvalue is a coefficient of proximity, (5.52) it indicates therefore that the distance in space  $L^2$  between  $F(t, \Omega)$  and nearest realized by function uncertainties/indeterminancies  $\chi(t, \Omega)$  are equal to zero. Consequently,  $F(t, \Omega)$  is realized as the function of uncertainty/indeterminacy.

## Chapter 6.

MAXIMUM AND MINIMUM OF THE PARTIAL SPACE OF THE BODY OF  
INDETERMINANCY.

## 6.1. Maximization of partial space.

In this chapter are examined the signals the functions of uncertainty/indeterminacy of which in a sense can be considered optimum.

As it was noted, for joint rangings and rate was desirable the "highly directional" function of uncertainty/indeterminacy, which has narrow central peak and it was equal to zero out of this peak. This function is impracticable, since the complete space of the body of uncertainty/indeterminacy does not depend on the structure of signal and is equal to unity, see (5.4). However, let us attempt to obtain some approximation/approach to this ideal form.

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Let us fix in the central plane  $(t, \Omega)$  the region  $\sigma$  (Fig. 6.1), which possesses central symmetry, i.e., if point  $(t, \Omega) \in \sigma$ , then

$(-\Delta, -\Omega) \in \sigma$ , and we will seek the signal the function of uncertainty/indeterminacy of which is maximally concentrated in the region  $\sigma$ , i.e., let us require so that the partial space of the body of uncertainty/indeterminacy, included in this field, would be maximum

$$V_{\sigma}(z) = \frac{1}{2\pi} \iint_{\sigma} |z(t, \Omega)|^2 dt d\Omega = \max. \quad (6.1)$$

Let us clarify this condition. In Chapter 2, examining signals with the maximum selectivity on the time, we attempted to bound signal by the assigned duration. When making these assumptions this proved to be impracticable, but one of the methods of approximation/approach was reduced to the maximization of the part of the energy, included in the assigned duration. Analogous property possess the optimum autocorrelation functions, examined in §2.5. Here, dealing in by the two-dimensional function of uncertainty/indeterminacy, we use a similar condition and we approach that so that the body of uncertainty/indeterminacy would be completely included within the region  $\sigma$ . Is accurate this impracticably, but the maximization of partial space (6.1) provides certain approximation/approach to this ideal. Since the complete space of the body of uncertainty/indeterminacy is fixed/recorded, automatically is provided the minimum of "energy" of remainders/residues out of the region  $\sigma$  <sup>1</sup>.

FOOTNOTE <sup>1</sup>. Let us emphasize that the formulated criterion of optimum

character assumes that within the region  $\sigma$  the targets can not be permitted, but is provided possible fully permission/resolution for two targets, which do not fall into one region  $\sigma$ . ENDFOOTNOTE.

The formulated problem borders on also following. In certain cases it is possible to indicate in the plane  $(t, Q)$  the region  $\sigma$ , in which should be expected the intense mixing reflections. For example, if the radar system, established/installed on the satellite, is intended for the detection of another satellite, then mixing will be reflections from suppress surfaces.

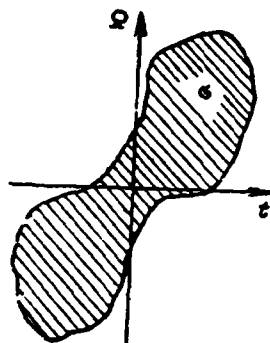


Fig. 6.1.

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Knowing trajectory and rate of a satellite-carrier, it is possible to indicate in the plane  $(t, \Omega)$  the region  $\sigma$ , which corresponds to the Doppler rates and the ranges of different sections suppress surfaces. Logical under these conditions to select signal so as to minimize the partial of the space of the body of uncertainty/indeterminacy in the region  $\sigma$ :

$$V_{\sigma}(z) = \iint_{\sigma} |X(t, \Omega)|^2 dt d\Omega = \min.$$

One of the problems of this kind is examined in the second part of this chapter (§§6.5-6.10).

## 6.2. Quasi-optimal signals.

Unfortunately, it is impossible to propose the straight/direct method of maximization (6.1). In this connection let us introduce further limitation to the class of the permissible signals and let us first somewhat change the formulation of the problem.

We will consider that signal  $s(t)$  can be either the even or odd function of time. As was noted, for such signals the functions of uncertainty/indeterminacy are real [70]. Maximum value  $\chi(0, 0) = 1$ , and, because of continuity, in certain vicinity of the central point  $t = \Omega = 0$  the function of uncertainty/indeterminacy is positive. We will seek signal  $s(t)$ , which maximizes value

$$W_s(s) = \frac{1}{2\pi} \iint \chi_s(t, \Omega) dt d\Omega = \max. \quad (6.2)$$

Logical to assume that the function of uncertainty/indeterminacy, which satisfies condition (6.2), is positive for all internal points of region  $\sigma$  (further this is confirmed based on example). Therefore the maximization of integral (6.2) is connected with the achievement of the highest possible positive values  $\chi_s(t, \Omega)$  within the region and, therefore, with an increase in partial space  $V_s(\sigma)$ , of that determined by formula (6.1). These considerations show that the signals, which satisfy condition (6.2), we will call their quasi-optimal - they are close to the optimum signals from the class of those permitted (even and odd).

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Important for future reference generalization is obtained, if we introduce real weight function  $g(t, \Omega)$  and to examine instead of (6.2) value

$$W_s(g, s) = \frac{1}{2\pi} \iint g(t, \Omega) \chi_s(t, \Omega) dt d\Omega. \quad (6.3)$$

The signals, which maximize this value, we will also call quasi-optimal, they depend not only on region  $\sigma$  but also on weight function  $g(t, \Omega)$ .

Before passing to the determination of optimum and quasi-optimal signals, let us note that during some strains of region  $\sigma$  the structure of the signals indicated is changed in an obvious manner.

Let us introduce instead of  $t$  and  $\Omega$  the dimensionless coordinates

$$\eta = t/\tau \text{ и } \xi = \Omega\tau,$$

where  $\tau$  - arbitrary scale time unit. It is easy to see that pi to this replacement of variable/alternating all previous relationships/ratios retain their form. Let in new coordinates  $(\eta, \xi)$  be selected the region  $\sigma$  and for it is determined signal  $s_0(\eta)$ , the

giving maximum to value

$$W_s(g, z) = \frac{1}{2\pi} \iint g(\tau, \xi) \chi_s(\tau, \xi) d\tau d\xi = \max.$$

If we return to coordinates  $t$  and  $\Omega$ , then it is obvious, a change in the scale  $\tau$  will lead to the arbitrary extension of region along the axis of time and its corresponding compression along the axis of frequency. Quasi-optimal signals are characterized by during this strain only scale and are given by the expression

$$s_1(t) = s_0(t/\tau).$$

Therefore the solution of assigned mission, for example, for the circle (this example is examined below) is easy to spread to the elliptical region, equivalent with the initial circle, if the principal axes of ellipse coincide with axes  $t$  and  $\Omega$ .

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Moreover, applying known theorem of Klauder [7, 38], it is possible to turn this ellipse to the arbitrary angle  $\theta$  (Fig. 6.2), after determining new quasi-optimal signal according to the formula

$$s_2(t) = \frac{1}{\tau} \sqrt{\frac{\operatorname{cosec} \theta}{2\pi}} \int_{-\infty}^{\infty} s_0\left(\frac{t'}{\tau}\right) \times \\ \times \exp \left\{ j \frac{t'^2 + t^2}{2\tau^2} \operatorname{ctg} \theta - \frac{tt'}{\tau^2} \operatorname{cosec} \theta \right\} dt'. \quad (6.4)$$

Thus, after determining optimum or quasi-optimal signals for some "base" regions, it is possible to considerably widen uses/applications due to the strains indicated.

Let us switch over to the determination of the quasi-optimal signal, which maximizes value (6.3). In view of the done assumptions weight function  $g(t, \Omega)$  and function uncertainties/indeterminancies  $\chi_s(t, \Omega)$  are real. Being congruent/equating (5.12) and (6.3) we see that in these assumptions value  $W_s(g, \sigma)$  is nothing else but the coefficient of proximity for functioning the uncertainty/indeterminancy  $\chi_s(t, \Omega)$  of the function

$$F(t, \Omega) = \begin{cases} g(t, \Omega) & \text{при } (t, \Omega) \in z; \\ 0 & \text{при } (t, \Omega) \notin z. \end{cases} \quad (6.5)$$

Key: (1). with.

Consequently, in accordance with the hypothesis of proximity the task of determining the quasi-optimal signal, which maximizes  $W_s(t, \Omega)$ , is equivalent to approximation/approach to function  $F(t, \Omega)$ , assigned in the form (6.5). It is possible to use the results of the previous chapter for the solution of this problem.

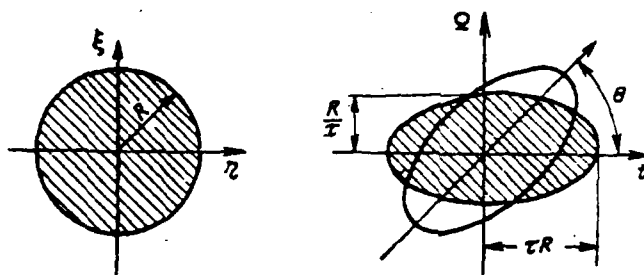


Fig. 6.2.

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In this case it is convenient to use the method of Sussman, which uses coordinate representations. As it was shown, solution gives eigenvector of matrix  $G$  whose elements/cells are determined in the form

$$G_{mn} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} g(t, \Omega) K_{nm}^*(t, \Omega) dt d\Omega, \quad (6.6)$$

and the maximum value  $W(g, \sigma)$ , attained at the quasi-optimal signal, is greatest eigenvalue  $\lambda_{max}$  of this matrix/die.

Here  $K_{nm}$  - derived base functions, connected with the selected set of base functions  $f_n(t)$  with relationship/ratio (5.23). It is noted also that function  $g(t, \Omega)$  possesses symmetry (5.17).

As is known, finding eigenvalues and eigenvectors of matrix is connected with the diagonalization. There is the unitary conversion

$$f_n^{(1)}(t) = \sum_m q_{mn} f_{mn}(t),$$

with which a base system  $f_n(t)$  it is converted into this new system

$f_n^{(1)}(t)$ , that the matrix/die  $G$  becomes diagonal, i.e.,

$$G_{nm} = \begin{cases} 0 & \text{при } m \neq n; \\ \lambda_n & \text{при } m = n. \end{cases}$$

Key: (1). with.

The elements/cells of principal diagonal are eigenvalues  $\lambda_n$  and new base functions  $f_n^{(1)}(t)$  are eigenvectors of matrix  $G$ . This method of finding of eigenvectors and eigenvalues not only and always not best, but in the task in question it leads to the necessary results.

### 6.3. Circular region.

Let the region be a circle of radius  $R$  1:

$$\rho^2 + \Omega^2 = r^2 \leq R^2.$$

FOOTNOTE 1. Here and throughout  $t$  and  $\Omega$  - dimensionless quantities. Such designations are used for simplification in the recording, it

would be it is more apply coordinates  $\eta = t/r$  and  $\xi = \Omega r$ . ENDFOOTNOTE.

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Let us select as the base functions in the space of the signals of the function of Hermite

$$f_n(t) = \frac{1}{(\sqrt{\pi} 2^n n!)^{1/2}} e^{-t^2/2} H_n(t),$$

where  $H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$  - Hermite's polynomial.

Then, for the derived base functions we obtain according to (5.23):

$$K_{nm}(t, \Omega) = (\pi 2^{n+m} n! m!)^{-1/2} \int_{-\infty}^{\infty} H_n\left(t' + \frac{t}{2}\right) \times \\ \times H_m\left(t' - \frac{t}{2}\right) \exp\left\{-\frac{(t' + t/2)^2 + (t' - t/2)^2}{2} + j\Omega t'\right\} dt'$$

or, after simple conversions,

$$K_{nm}(t, \Omega) = \frac{e^{-(t^2 + \Omega^2)/4}}{(2\pi^{n+m} n! m!)^{1/2}} \int_{-\infty}^{\infty} e^{-x^2} \times \\ \times H_n\left(x + \frac{t + j\Omega}{2}\right) H_m\left(x - \frac{t - j\Omega}{2}\right) dx.$$

The value of latter/last integral is known ([21], page 852). After using polar coordinates  $(r, \phi)$  in the plane  $(t, \Omega)$ , finally we find:

$$K_{nm}(r, \phi) = \rho_{nm}(r) e^{j(n-m)\phi}, \quad (6.7)$$

where

$$p_{nm}(r) = \begin{cases} \left( \frac{n!}{m! 2^{n-m}} \right)^{1/2} r^{n-m} L_m^{(n-m)} \left( \frac{r^2}{2} \right) e^{-r^2/4}; & n \geq m, \\ \left( \frac{m!}{n! 2^{m-n}} \right)^{1/2} r^{m-n} L_n^{(m-n)} \left( \frac{r^2}{2} \right) e^{-r^2/4}; & n < m. \end{cases}$$

Here  $L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha})$  Laguerre's polynomial<sup>1</sup>.

FOOTNOTE 1. These results are of interest also in the following sense. As is known, the functions of Laguerre  $e^{-x/2} x^{\alpha/2} L_n^{(\alpha)}(x)$  form complete orthogonal system in interval  $(0, \infty)$ . Taking into account the completeness of the system of harmonic functions  $e^{in\varphi}$  in the interval  $(0, 2\pi)$  it is not difficult to note that functions (6.7) form complete system in the entire plane. Any other system of derived base functions is connected with functions (6.7) with unitary conversion. This proves the completeness of the system of derived bases, by any generated complete system of orthogonal functions in the space of signals. ENDFOOTNOTE.

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Let us now count matrix elements  $G$ , by assuming that weight function possesses the circular symmetry:  $g(t, \varphi) = g(r)$ . Passing in integral (6.6) to the polar coordinates, we find

$$G_{nm} = \frac{1}{2\pi} \int_0^R g(r) p_{nm}(r) r dr \int_0^{2\pi} e^{-j(n-m)\varphi} d\varphi. \quad (6.8)$$

Is obvious,  $G_{nm}=0$  with  $m \neq n$  and

$$\begin{aligned} G_{nn} = \lambda_n &= \int_0^R g(r) p_{nn}(r) r dr = \int_0^R g(r) e^{-r^2/4} L_n\left(\frac{r^2}{2}\right) r dr = \\ &= \int_0^{R^2/2} g(\sqrt{2x}) e^{-x/2} L_n(x) dx. \end{aligned} \quad (6.9)$$

Consequently,  $G$  - diagonal matrix/die. This means that after selecting for the function of Hermite, we "guessed" that only system of base functions in the space of signals, for which matrix/die  $G$  was diagonal, this occurring for any weight function, symmetrical on  $\phi$ .

From previous it follows that the base functions of this system - the function of Hermite

$$s_n(t) = \frac{1}{(\sqrt{\pi} 2^n n!)^{1/2}} e^{-t^2/2} H(t) \quad (6.10)$$

are the signals for which the partial space  $W(g, \sigma)$  takes outer limits.

Outer limits themselves are equal to eigenvalues  $\lambda_n$  and they are expressed by integral (6.9), and, although eigenvalues depend on weight function, extreme signals - eigenvectors of matrix - the same for weight functions, which possess circular symmetry.

In order to find the quasi-optimal signal, which maximizes  $W(g, \sigma)$ , it suffices to now select greatest of the extrema, greatest

eigenvalue  $\lambda_n$ . As it is clear from (6.9), values  $\lambda_n$  are coefficients of the expansion of the function

$$F(\sqrt{2}x) = \begin{cases} g(\sqrt{2}x) & \text{при } x < \frac{R^2}{2} \\ 0 & \text{при } x > \frac{R^2}{2} \end{cases} \quad (1)$$

Key: (1). with.

according to the functions of Laguerre.

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From the character of the latter it is clear that if function  $g(r)$  is positive and mono-tone decreases (it does not grow) with increase in  $r$ , then maximum  $\lambda_n$  is obtained for  $n=0$ . Therefore for any decreasing weight function quasi-optimal is the Gaussian signal

$$s_0(t) = \pi^{-1/4} e^{-t^2/2} \quad (6.11)$$

for which

$$\chi_s(t, \Omega) = e^{-(t^2 + \Omega^2)/4} = e^{-r^2/4} \quad (6.12)$$

Consequently, setting  $g=1$ , we have from (6.9)

$$\max W_s(\tau) = \lambda_0 = \int_0^R e^{-r^2/4} r dr = 2(1 - e^{-R^2/4}).$$

For weight functions of another type, which have an oscillatory nature, quasi-optimal signal can be another function of Hermite, but these cases us further interest will not be.

## 6.4. Optimum signals.

Let us return to the task about the optimum signal, which maximizes partial space  $V_*(\sigma)$ , determined by formula (6.1). Without having the capability to propose the direct method of determining this signal, let us construct the iterative process, which leads, in the limit, to the solution. Being limited to the again real functions of uncertainty/indeterminacy (even and odd signals), let us introduce into the examination value

$$W_{sh}(\sigma) = \frac{1}{2\pi} \iint \chi_s(t, \Omega) \chi_h(t, \Omega) dt d\Omega, \quad (6.13)$$

where  $\chi_s$  and  $\chi_h$  - functions of the uncertainty/indeterminacy of signals  $s(t)$  and  $h(t)$  respectively, and we will increase this value as follows.

Let us assign first certain signal  $s^{(0)}(t)$  and after defining function  $\chi_s^{(0)}(t, \Omega)$ , maximize value  $W_{sh}(\sigma)$ , by selecting signal  $h^{(0)}(t)$  and by considering function  $\chi_s^{(0)}(t, \Omega)$  as weight

$$\chi_s^{(0)}(t, \Omega) = g(t, \Omega).$$

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As we saw, for this it is necessary to determine from formula (6.6) matrix/die  $G^{(0)}$  and to find its eigenvector, which corresponds to maximum eigenvalue  $\lambda_{max}^{(0)}$ . Maximum value  $W_{sh}(\sigma)$  is equal to this

eigenvalue

$$\max_h W_{sh}^{(0)}(\sigma) = \lambda_{max}^{(0)}.$$

Then, fixing/recording the obtained signal  $h^{(0)}(t)$  and the function of uncertainty/indeterminacy  $\chi_h^{(0)}(t, \Omega)$ , it is determined the quasi-optimal signal  $s^{(1)}(t)$ , maximizing integral (6.13), in which now weight function is  $\chi_h^{(0)}(t, \Omega)$ . Appropriate eigenvalue let us designate  $\lambda_{max}^{(1)}$ .

The following approximations/approaches are obtained analogously: with fixed/recorded  $s^{(1)}(t)$  is determined the quasi-optimal signal  $h^{(1)}(t)$ , and then, on the contrary, on signal  $h^{(1)}(t)$  is determined the signal of the second approximation/approach  $s^{(2)}(t)$  and so forth. It is easy to see that this process leads to the ascending series of eigenvalues

$$\lambda_{max}^{(0)} < \lambda_{max}^{(1)} < \dots \quad (6.14)$$

This sequence is bounded above. Actually/really, as it is clear from (6.13),  $W_{sh}(\sigma)$  is a coefficient of the proximity of functions  $\chi_s(t, \Omega)$  and  $\chi_h(t, \Omega)$ . Consequently,  $\lambda_{max} \leq 1$ .

The aforesaid means that sequence (6.14) converge to certain limit. As a result are determined two saturation signals  $s_{opt}(t)$  and  $h_{opt}(t)$ , for which value  $W_{sh}(\sigma)$  takes greatest (limiting) value. Let us demonstrate that without taking into account unessential phase factor these saturation signals coincide  $h_{opt}(t) = s_{opt}(t)$  and, therefore,

iterations lead to the maximum value

$$V_s(\sigma) = W_{ss}(\sigma) = \frac{1}{2\pi} \int \int \chi_s^2(t, \Omega) dt d\Omega.$$

Let for the selected region  $\sigma$  there be one or several optimum functions of uncertainty/indeterminacy  $\hat{\chi}_k(t, \Omega)$ , for each of which partial space  $V_s(\sigma)$  it has the greatest possible value of  $V_0$ :

$$\max_s V_s(\sigma) = \max_s \frac{1}{2\pi} \int \int \chi_s^2 dt d\Omega = \frac{1}{2\pi} \int \int \hat{\chi}_k^2 dt d\Omega = V_0(\sigma).$$

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Then, applying to (6.13) Schwarz-Buniakowski's inequality, we can register

$$\begin{aligned} W_{sh}^2(\sigma) &\leq \frac{1}{2\pi} \int \int \chi_s^2(t, \Omega) dt d\Omega \times \\ &\times \frac{1}{2\pi} \int \int \chi_h^2(t, \Omega) dt d\Omega = V_s(\sigma) V_h(\sigma) \leq V_0^2(\sigma). \end{aligned} \quad (6.15)$$

If will be achieved/reached equality in both of these inequalities, value  $W_{sh}(\sigma)$  will take greatest possible value. But in the relationship/ratio of a Schwarz-Buniakowski equality is reached only in such a case, when factors are proportional. Taking into account standardization, this means that the functions of uncertainty/indeterminacy must coincide. As a result in order to ensure equality also in the latter from relationships/ratios (6.15), it is necessary to satisfy the condition

$$\chi_s(t, \Omega) = \chi_h(t, \Omega) = \hat{\chi}_k(t, \Omega). \quad (6.16)$$

The function of uncertainty/indeterminacy uniquely determines

the realizing signal, if we do not consider the arbitrary initial phase (see §7.1). Therefore without taking into account this phase condition (6.16) is equivalent to the following:

$$s_{opt} = h_{opt}(t) = \hat{s}_k(t),$$

where  $\hat{s}_k(t)$  the signal, which realizes the optimum function of uncertainty/indeterminacy  $\hat{\gamma}_k(t, \Omega)$ .

Thus, when iterations actually/really give global maximum to value  $W_{sh}(\sigma)$ , they lead to two identical signals which are determined as a result of approximations/approaches.

Relative to the uses/applications of this method it is possible to note the following. It is substantial, what signal  $s^{(0)}(t)$  is selected as the initial on first stage. The nearer this signal to the optimum, the more rapid the iteration they lead to the target. Furthermore, the unsuccessful selection of initial approximation/approach can lead to the erroneous result: will be found the maximum of value  $W_{sh}(\sigma)$ , but not greatest, global. Iterations one should begin from the signal, close to the optimum, and as similar it is expedient to select the quasi-optimal signal, which maximizes value (6.2). As it was noted, the quasi-optimal signals are close and optimum.

The method examined greatly easily leads to the result for the circular region  $\sigma$ . After selecting as zero approximation the quasi-optimal Gaussian signal (6.11) with the function of uncertainty/indeterminacy (6.12), it is necessary during the first stage to determine signal  $h^{(0)}(t)$ , that maximizes value

$$W_{sh}(z) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^R e^{-r^2/4} \chi_h^{(0)}(r, \varphi) r dr d\varphi.$$

Since weight function  $g(r, \varphi) = e^{-r^2/4}$  depends only on  $r$  and monotonically it decreases, as shown in §6.3, signal  $h^{(0)}(t)$  will be Gaussian -

$$h^{(0)}(t) = \pi^{-1/4} e^{-t^2/2}.$$

It is obvious, further approximations/approaches will also give Gaussian signals in each stage. Thus, for the circular region we again "guessed" optimum signal - this is the Gaussian signal, which coincides with the quasi-optimal.

Knowing optimum signal, it is easy to count the maximum partial space of the body of uncertainty/indeterminacy for the circular region:

$$\begin{aligned} \max_s V_s(z) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^R \chi_{opt}^2(t, \Omega) dt d\Omega = \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^R e^{-r^2/2} r dr = 1 - e^{-R^2/2}. \end{aligned} \quad (6.17)$$

If, using the formula of Klauder (6.4) to deform circular region into the elliptical with the arbitrary inclination/slope, then optimum will be in general ChM impulses/momenta/pulses with a linear change in the frequency and gaussian envelope [7]:

$$s(t) = \exp \left\{ - \left( \frac{1}{\sigma^2} - j\gamma \right) \frac{t^2}{2} \right\}.$$

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But in accordance with that presented the value of partial space during this strain is not changed and it is as before given by formula (6.17), which is conveniently registered in the form

$$\max V_s(z) = 1 - e^{-S/2\sigma^2}, \quad (6.18)$$

where  $S$  - area of ellipse in the plane  $(t, \Omega)$ . This result can be used for the evaluation/estimate of maximum partial space in other, not elliptical regions. It is easy to show that if certain region  $\sigma_1$  is wholly included within  $\sigma_2$ , then  $V_{\max}(\sigma_1) < V_{\max}(\sigma_2)$ . Therefore, if the assigned arbitrary region  $\sigma$  is described by certain ellipse with an area of  $S_1$  and is inscribed in it another ellipse with an area of  $S_2$ , then on the basis (6.18) we obtain

$$1 - e^{-S_2/2\sigma^2} > \max V_s(z) > 1 - e^{-S_1/2\sigma^2}. \quad (6.19)$$

Apparently, this relationship/ratio sufficiently fully considers maximum partial space for the regions, which are of practical interest.

As showed Klauder [38] hermitian signals (6.10) are only, for

which the functions of uncertainty/indeterminacy possess circular symmetry. It is logical to assume that the maximization of the partial space of the body of uncertainty/indeterminacy in the circular region requires a similar symmetry. A strict proof of this position is given above [9].

The global maximum of partial space gives Gaussian signal - zero function of Hermite. For this signal the contraction coefficient is of the order of one. One of the ways of transition/junction to the signals with the large compression is connected with the strain of region into the elliptical. In this case is obtained LFM impulse/momentum/pulse with gaussian envelope. Another possibility consists in the fact that after preserving circular region we are given contraction coefficient, but we seek global, but one of local of the maximums of partial space. As can be seen from that presented, in this case should be selected the function of Hermite of high order, moreover order must be matched with the assigned compression.

Subsequently will be shown that this path it leads to the signals whose satisfactory approximation give signals with the phase manipulation. Thus, the task about the maximum of the partial space of the body of uncertainty/indeterminacy leads to two most widely used classes of serrated signals - ChM and FM.

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The maximum partial space of the body of uncertainty/indeterminacy was traced also in the work of Price and Hofstetter [55]. In this work are obtained useful evaluations/estimates for the maximum partial space, but are not revealed the signals, which realize this space.

#### 6.5. Minimization of partial space in the assigned interference zone.

To the minimization of the partial space of the body of uncertainty/indeterminacy leads the task about the decrease (or complete suppression) of the mixing reflections from the distributed in the space multiple reflectors - dipole cloud, the underlying terrestrial or sea surface, etc.

Let in the plane  $(t, \Omega)$  be assigned the interference zone  $\sigma$ , i.e., the region, in which are concentrated the mixing reflections, and with  $t=\Omega=0$  is located the observed pinpoint target. Signal amplitude from this target is proportional to the value of the function of uncertainty/indeterminacy  $|x(0,0)|$ , and the average/mean power of passive jamming - to value of the partial space  $V(\sigma)$ . If we take into account also the inherent noise of receiver, the ratio "signal/(interference + noise)" obtains expression (with an accuracy

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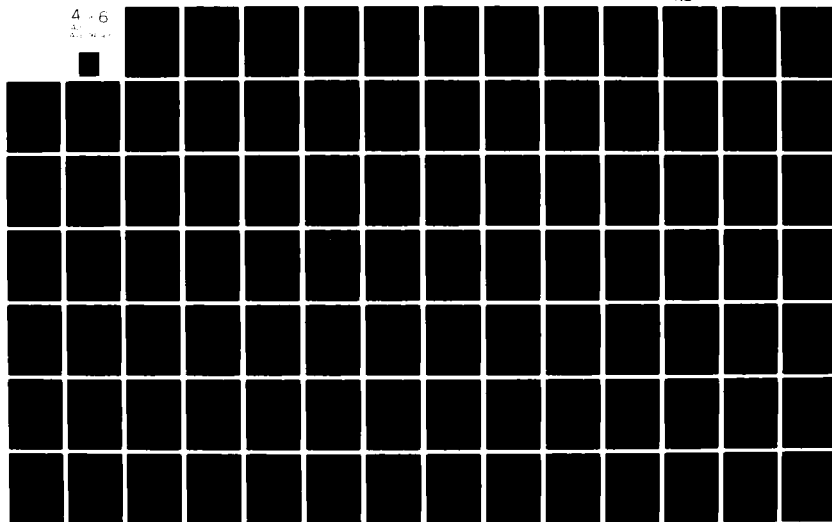
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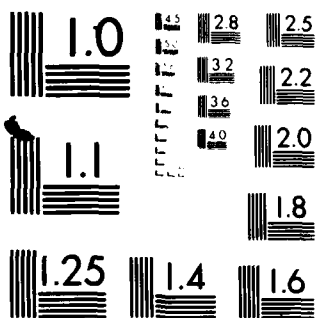
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to unessential for us constant factor)

$$\frac{C}{\pi + \beta} = \frac{U^2}{V(\sigma) + \beta}, \quad U = |\chi(0, 0)|. \quad (6.20)$$

where  $\beta$  - constant, which depends on the relationship/ratio of the noise density and specific jamming intensity.

Therefore the maximization of the excess of the signal above the interferences is reduced to the task about the minimum of the functional

$$\Phi = V(\sigma) - \mu U = \min, \quad (6.21)$$

where  $\mu$  - indefinite factor of Lagrange. This task has great practical value, and by it is given much attention [58, 62, 69, 71, 89, 91, 92, etc.].

Not always the maximum of the excess of the signal above the interferences is achieved by the agreement of the sounding signal and filter. Therefore into the space the case it is necessary to examine the cross function of uncertainty/indeterminacy  $\chi_{sh}(t, \Omega)$  and the partial space

$$V_{sh}(\sigma) = \frac{1}{2\pi} \int \int |\chi_{sh}(t, \Omega)|^2 dt d\Omega.$$

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In this connection are studied three tasks:

1) the optimization of filter (receiver)  $h(t)$  with the assigned signal  $s(t)$ , i.e., the task about the optimum mismatched processing;

2) optimization of the pair signal-filter, i.e., determination  $s(t)$  and  $h(t)$ , with which functional (6.21) reaches the global minimum;

3) the optimization of signal  $s(t)$  during the matched processing, i.e., with  $s(t)=h(t)$ .

Let us note that in the latter case  $U=|\chi(0,0)|=1$  and minimization (6.21) is reduced to the "pure/clean" task about the minimum of partial space.

It is not difficult to note also that in the case of the mismatched processing the partial space is a quadratic functional relative to  $h(t)$  and relative to  $s(t)$ , and during the matched processing  $V$  there is a functional of the fourth degree relative to  $s(t)$ . Therefore the first of the tasks indicated are substantially simpler than the others. But even this task is reduced in the space the case, to the problem of eigenvalues, and finding of the corresponding eigenfunction (optimum characteristic of filter) it requires bulky calculations<sup>1</sup>.

FOOTNOTE 1. Let us note the close analogy of this task with the maximization of the partial space where we succeeded in only "guessing" the solution in the particular case, and also with the synthesis of the functions of uncertainty/indeterminacy, examined in Chapter 5. ENDFOOTNOTE.

The synthesis of optimum pair signal-filter and task about the optimum matched processing is even more complicated. The general efficient methods of their solution apparently there does not exist.

So is matter, if interference zone  $\omega$  is arbitrary. We will examine interference zone in the form of infinite Doppler band (Fig. 6.3). This zone is of practical interest for the Doppler systems and in some analogous cases. The zone of this form was traced in [92], but the work indicated contains faster the formulation of the problem, than its solution.

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Meanwhile as it will be shown, for the zone of this form it is possible to obtain the comprehensive analytical solutions of all three tasks - about the optimum mismatched processing, about the optimum pair signal-filter indicated and about the optimum of signal with matched filter. We will see also, that these tasks, although

they examine the function of uncertainty/indeterminacy, are close to the synthesis of correlation functions, traced in Chapter 4. In particular, the solution significantly uses Gibbs's lemma (see §4.6) and its generalization, given below.

Let us note also that after using the theorem of Klauder (6.4), it is possible to place interference band at arbitrary angle in the plane  $(t, \Omega)$  and by the fact to somewhat widen the field of application of our results.

#### 6.6. Interference zone in the form of infinite Doppler band.

For the interference zone in question the partial space of the body of uncertainty/indeterminacy has a value

$$V = \frac{1}{2\pi} \int_{-\Delta}^{\nu+\Delta} d\Omega \int_{-\infty}^{\infty} |\chi_{sh}(t, \Omega)|^2 dt = \frac{1}{2\pi} \int_{-\Delta}^{\nu+\Delta} v(\Omega) d\Omega, \quad (6.22)$$

where  $v(\Omega)$  - "space on lines".

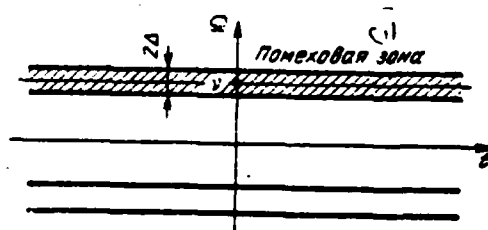


Fig. 6.3.

Key: (1). Interference zone.

Using a representation of the function of the uncertainty/indeterminacy through the spectra, we obtain

$$\begin{aligned}
 v(\Omega) &= \int_{-\infty}^{\infty} |\chi_{sh}(t, \Omega)|^2 dt = \\
 &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{s}\left(\omega - \frac{\Omega}{2}\right) \tilde{h}^*\left(\omega + \frac{\Omega}{2}\right) \tilde{s}^*\left(\omega' - \frac{\Omega}{2}\right) \times \\
 &\quad \times \tilde{h}\left(\omega' + \frac{\Omega}{2}\right) \int_{-\infty}^{\infty} e^{it(\omega - \omega')} dt d\omega' d\omega = \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{s}\left(\omega - \frac{\Omega}{2}\right) \tilde{h}^*\left(\omega + \frac{\Omega}{2}\right) \tilde{s}^*\left(\omega' - \frac{\Omega}{2}\right) \times \\
 &\quad \times \tilde{h}\left(\omega' + \frac{\Omega}{2}\right) \delta(\omega - \omega') d\omega d\omega' = \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \tilde{s}\left(\omega - \frac{\Omega}{2}\right) \right|^2 \left| \tilde{h}\left(\omega + \frac{\Omega}{2}\right) \right|^2 d\omega = \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} a^2(\omega) b^2(\omega + \Omega) d\omega, \quad (6.23)
 \end{aligned}$$

where  $a(\omega)$  and  $b(\omega)$  - the amplitude spectra, which correspond to signal and filter.

As with the synthesis of correlation functions, we will be bounded to the class of the even amplitude spectra  $a(\omega)$  and  $b(\omega)$  (see §4.1). Then from (6.23) we have

$$\begin{aligned}
 v(-\Omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} a^2(\omega) b^2(\omega - \Omega) d\omega = \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} a^2(-\omega) b^2(-\omega - \Omega) d\omega = \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} a^2(\omega) b^2(\omega + \Omega) d\omega = v(\Omega).
 \end{aligned}$$

From this relationship/ratio it is clear that, without changing the value of partial space, it is possible to supplement to the interference zone in question the symmetrical band, shown in Fig.

6.3. Therefore we will use instead of (6.22) the expression

$$V = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\Omega) |\chi(t, \Omega)|^2 dt d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\Omega) \sigma(\Omega) d\Omega,$$

where  $Q(\Omega)$  - even weight function of the form

$$Q(\Omega) = \begin{cases} 1/2, & \text{при } \nu - \Delta < |\Omega| < \nu + \Delta; \\ 0, & \text{в остальных случаях.} \end{cases} \quad (6.24)$$

Key: (1). with. (2). in remaining cases.

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Taking into account (6.23), is easy to lead the expression of particular space to the canonical bilinear form relative to  $a^2(\omega)$  and  $b^2(\omega)$ :

$$V = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(\omega - \omega') a^2(\omega') b^2(\omega) d\omega d\omega'. \quad (6.25)$$

In view of  $Q(\Omega)$  the kernel of this form is symmetrical<sup>1</sup>.

FOOTNOTE <sup>1</sup>. The symmetry of kernel is caused by the fact that we examine only the even amplitude spectra. But this simplifying limitation does not have fundamental value. Further results are valid and in general. ENDFOOTNOTE.

We see that the particular space in the band does not depend on the phase spectra of signal and filter. This leads in the final analysis to the generality with the task of the synthesis of correlation functions. On the other hand, value

$$U = |\gamma_{sh}(0, 0)| = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}(\omega) \tilde{h}^*(\omega) d\omega \right|,$$

it is obvious, it depends on phase spectra. However, with any assigned  $a(\omega)$  and  $b(\omega)$  maximum  $U$  takes the place when phase spectra are matched, i.e., when

$$\arg \tilde{s}(\omega) = \arg \tilde{h}(\omega), \quad (6.26)$$

$$U = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b(\omega) d\omega. \quad (6.27)$$

Therefore the maximization of relation signal/noise (6.20) is reduced for our zone to variational problem (6.21), moreover  $V$  and  $U$  are determined according to (6.25) and (6.27). In this case the function of uncertainty/indeterminacy must, of course, be normalized, which indicates the standardization of energy of signal and filter:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} a^2(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} b^2(\omega) d\omega = 1. \quad (6.28)$$

In the case of the assigned signal (task 1) the amplitude spectrum  $a(\omega)$  is fixed/recorded and should be sought only the characteristic of filter  $b(\omega)$ . During the optimization of the pair signal-filter (task 2) are found out both functions  $a(\omega)$  and  $b(\omega)$ .

Finally, for the optimization of signal with the matched filter (task 3) should be to place  $b(\omega)=a(\omega)$  and  $U=1$  and minimized space  $V$ , selecting  $a(\omega)$ .

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In all tasks the phase spectra remain arbitrary (with the fulfillment of agreement (6.26)). This means that there are many optimum signals, which are characterized by phase spectra, and it is possible in the latter/last stage of synthesis to select the signal, most convenient for the practical realization.

It is not difficult to be convinced however that without the further conditions the formulated task has a series/row of trivial solutions and is not of interest. Actually/Really, as we now will show, retaining maximally possible value of  $U=1$ , always it is possible to obtain arbitrarily low partial space, including  $V=0$ , moreover even in this case the solution is not singular.

As it follows of (6.27) and (6.28), value  $U=1$  is always achieved by the matched processing, i.e., with  $a(\omega)=b(\omega)$ . Assuming this condition carried out, let us consider first the case of narrow interference band, i.e.,  $\Delta \rightarrow 0$ . Then  $V$  is space on the line and according to (6.23)

$$V = \frac{\Delta}{\pi} \int a^2(\omega) a^2(\omega - \nu) d\omega.$$

Fig. 6.4a shows the spectrum of the periodic structure with which this space is equal to zero, since  $a(\omega)$  and  $a(\omega - \nu)$  nowhere overlap. In general, with the final width of zone  $\Delta$ , it is necessary to only respectively widen zero regions in the spectrum, as shown in Fig. 6.4b.

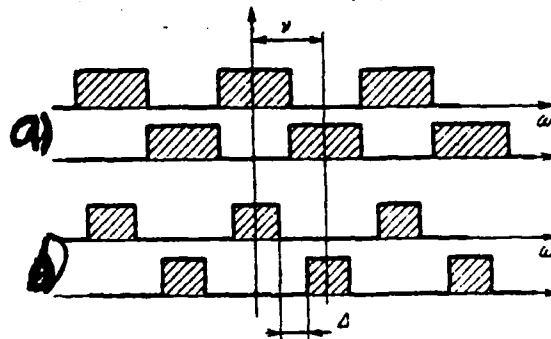


Fig. 6.4.

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Then for each value  $\Omega$  from the interference zone space on the line is equal to zero. Let us note also that in the nonzero regions of the spectrum can have arbitrary form, provided these regions did not overlap with the appropriate shifts/shears. Furthermore, it is possible, of course, to vary the span of the spectrum, changing a number of nonzero zones. Let us note that the particular spectra of this form correspond to the signals, characteristic for Doppler RLS, to monochromatic oscillation/vibration and to coherent packet.

So that our task would become meaningful, it is necessary to superimpose further limitation on amplitude spectra  $a(\omega)$  and  $b(\omega)$  and, therefore, for the correlation function

$$R(t) = \chi(t, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b(\omega) e^{i\omega t} d\omega.$$

But even limitation of the duration of correlation function (parameter of permission/resolution according to Woodward) is here insufficient, since this duration depends in essence on the overall width of the spectrum which, as we saw, was arbitrary.

The practical limitation of necessary form is obtained from the following consideration. The periodic spectra of the type Fig. 6.4 give the multipeak correlation functions  $R(t)$  (cf. the case of coherent packet). From previous it is clear that with this correlation function it is possible to obtain zero interference level (if only interference zone does not switch on axis  $\Omega=0$ ). But in multipeak  $R(t)$  is essential the ambiguity of the measurement of the time of arrival. Therefore should be sought a compromise between the interference level and the desired form  $R(t)$ . For this we will bound amount of deflection

$$\begin{aligned} W &= \int_{-\infty}^{\infty} |R(t) - F(t)|^2 dt = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [a(\omega) b(\omega) - \tilde{F}(\omega)]^2 d\omega = \text{const.} \end{aligned} \quad (6.29)$$

where  $F(t)$  - the desired (single-peak) is correlation function, and  $\tilde{F}(\omega)$  - its spectrum.

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Let us note that  $F(t)$  can be unrealizable and with respect to  $\tilde{F}(\omega)$  can take negative values.

Taking into account (6.29) we come to the minimization of the functional

$$\Phi = V - \mu U - cW, \quad (6.30)$$

where  $c$  - new arbitrary factor. In the case of the matched processing  $U=1$  and corresponding component/term/addend in (6.30) must be excluded.

6.7. Optimization is filter with the assigned signal.

Let us pass to the solution of the first of the tasks in question, problem about the optimum mismatched processing. We will consider that the spectrum of signal  $a(\omega)$  is assigned, and we minimize (6.30), selecting the spectrum of filter  $b(\omega)$ . After rewriting (6.25) in the form

$$V = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\omega) b^2(\omega) d\omega,$$

where

$$K(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega - \omega') a^2(\omega') d\omega', \quad (6.31)$$

taking into account (6.27) and (6.29) let us present the functional being investigated in the form

$$\Phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{K(\omega)g(\omega) - \mu a(\omega)\} \overline{g(\omega)} - \\ - c[a(\omega)\overline{g(\omega)} - \tilde{F}(\omega)]^2 d\omega = \min.$$

moreover the unknown function  $g(\omega) = b^2(\omega)$  it is subordinated to the limitations

$$g(\omega) \geq 0 \text{ и } \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) d\omega = 1. \quad (6.32)$$

The second of these limitations corresponds to standardization (6.28).

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In view of Gibbs's lemma (see §6.9) optimum  $g(\omega)$  it satisfies the condition

$$K(\omega) - ca^2(\omega) - (\mu/2 - c\tilde{F}(\omega)) \frac{a(\omega)}{\sqrt{g(\omega)}} = \lambda \frac{a'(\omega)}{\sqrt{g(\omega)}} > 0, \\ \frac{a(\omega)}{\sqrt{g(\omega)}} = \lambda \frac{a'(\omega)}{\sqrt{g(\omega)}} = 0.$$

Key: (1). with.

where  $\lambda$  - certain constant. Therefore

$$b(\omega) = \frac{(\mu/2 - c\tilde{F}(\omega)) a(\omega)}{K(\omega) - ca^2(\omega) - \lambda} > 0, \quad (6.33)$$

moreover for those  $\omega$ , where the latter/last inequality is not fulfilled, should be placed  $b(\omega) = 0$ .

The parameter  $\lambda$  in (6.33) must be fitted so as to satisfy standardization (6.32), and the parameters  $\mu$  and  $c$  so as to fulfill assigned  $U$  and  $W$ . The determination of these parameters is connected with some difficulties. But, being given  $\mu$  and  $c$  (and selecting  $\lambda$  from standardization condition), it is possible to construct biparametric family of curves  $b(\omega) = b(\omega; \mu, c)$ , which minimize partial space with different  $U$  and  $W$ . According to (6.33) with the arbitrary

signal optimum filter with it is not matched ( $b(\omega) \neq a(\omega)$ ). After assuming in (6.34)  $c=0$ , we come to the decision of task (6.21) without further condition (6.29). In the case of the assigned signal this task makes sense.

#### 6.8. Optimization vapors signal - filter.

Let us consider first the task, reverse/inverse of previous. Let the spectrum of filter  $b(\omega)$  be assigned and is required to fit the optimum spectrum of signal  $a(\omega)$  in order to minimize functional (6.30). In view of the complete symmetry of these tasks not difficult comprehending that unknown spectrum is determined from the same formulas (6.31) and (6.33), if we in them interchange the position  $a(\omega)$  and  $b(\omega)$ , i.e.,

$$a(\omega) = \frac{(\mu^2 - c^2 \bar{F}(\omega)) b(\omega)}{K(\omega) - c b^2(\omega) - \lambda} > 0. \quad (6.34)$$

where

$$K(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega - \omega') b^2(\omega') d\omega'.$$

the parameters  $\lambda$ ,  $\mu$  and  $c$  are defined, as it is earlier. It is here taken into consideration, that kernel  $Q(\omega - \omega')$  is symmetrical.

Using these results let us show that the optimization of the pair signal-filter leads in our case to the matched processing. Actually/really, being transmitted from the suitable initial approximation/approach  $a_0(\omega)$ , it is possible to construct the following iterative process. First through the spectrum of signal  $a_0(\omega)$  we find, using formula (6.33), optimum for this signal spectrum of filter  $b_0(\omega)$ . Then through  $b_0(\omega)$  we find through formula (6.34) optimum for this filter spectrum of signal  $a_1(\omega)$ , and, continuing, we determine  $b_n$  on  $a_n$  and  $a_{n+1}$  on  $b_n$ .

Since at each space (during the definition of signal from the filter or filter on the signal) is used, in essence, one and the same formula, the process in question can be treated as the process of determining not two functions  $a(\omega)$  and  $b(\omega)$ , but one function  $z(\omega)$ , the algorithm of successive approximations taking the form

$$z_{i+1}(\omega) = \frac{(\mu^2 - cF(\omega))z_i(\omega)}{K_i(\omega) - cz_i^2(\omega) - \lambda} > 0. \quad (6.35)$$

Values  $z_i$  with the even numbers  $i=2n$  give successive approximations  $a_n$  and with the odd  $i=2n+1$  - value  $b_n$ .

If this process descends, values  $z_i$  and  $z_{i-1}$  will converge and in the limit they will coincide. We will arrive, therefore, at the agreement of signal and filter:  $a(\omega)=b(\omega)$ . However, the convergence of process (6.35) is not proved. Therefore we will use another

consideration.

If pair signal-filter is optimum, then signal is optimal for the filter, and filter - for the signal. In other words,  $a(\omega)$  and  $b(\omega)$  simultaneously satisfy conditions (6.33) and (6.34). From (6.33) it is evident that if with certain  $\omega$  spectrum  $a(\omega)=0$ , then also  $b(\omega)=0$ , while from (6.34) follows reverse/inverse confirmation. Therefore the frequency domains, in which  $a(\omega) \neq 0$  and  $b(\omega) \neq 0$  coincide. We will further examine only this frequency region, i.e., to assume/set  $a(\omega) > 0$  and  $b(\omega) > 0$ . Let us introduce the designation

$$L(g) = \frac{1}{2\pi} \int Q(\omega - \omega') g(\omega') d\omega' - cg(\omega).$$

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Then, combining (6.33) and (6.34), it is not difficult to obtain

$$\frac{L(a^2) - \lambda_1}{a^2} = \frac{L(b^2) - \lambda_2}{b^2} = \varphi.$$

where  $\varphi = \varphi(\omega)$  - certain unknown function. Thus,  $a^2(\omega)$  and  $b^2(\omega)$  are the decisions of the linear integral equation

$$L(g) - \varphi g = \frac{1}{2\pi} \int Q(\omega - \omega') g(\omega') d\omega' - \varphi(\omega) g(\omega) - cg(\omega) = \lambda. \quad (6.36)$$

which is not difficult to reduce to the equation of Fredholm of the second order in the standard form. Let us note that the constant  $\lambda$  can be different for  $a^2(\omega)$  and for  $b^2(\omega)$ , but function  $\varphi(\omega)$  and

operator  $L$  the same.

In view of Fredholm's alternative with any function  $\phi(\omega)$  the decision of equation (6.36) is singular, if it exists (excluding, perhaps, certain multitude of values  $c$  of zero measure). If decision exists, it linearly depends on  $\lambda$  (since the equation is linear). Consequently, spectra  $a(\omega)$  and  $b(\omega)$  are proportional to each other:

$$\frac{a^2(\omega)}{\lambda_1} = \frac{b^2(\omega)}{\lambda_2},$$

or, considering standardization (6.28),

$$a(\omega) = b(\omega). \quad (6.37)$$

We come to the conclusion that for the interference zone in question the task of synthesis of the pair signal-filter does not have independent value. After solving this task, we will not obtain the best results now during the matched processing, i.e., by examining condition (6.37) as further limitation.

#### 6.9. Generalization of Gibbs's lemma.

Passing to the task about the optimum matched treatment, we assume/set  $a^2(\omega) = b^2(\omega) = g(\omega)$  and  $U=1$ . Then fundamental functional (6.30) takes the form

$$\Phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{K(\omega)g(\omega) - c[g(\omega) - \tilde{F}(\omega)]^2\} d\omega = \min. \quad (6.38)$$

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The unknown function  $g(\omega)$  is as before subordinated to limitations (6.32), but in contrast to the previous tasks now  $K(\omega)$  it depends on  $g(\omega)$ :

$$K(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega - \omega') g(\omega') d\omega'. \quad (6.39)$$

This fact does not permit us to use for the minimization Gibbs's lemma, formulated in §4.6 and used above, but, as it will be shown, decision can be constructed on the basis of the following generalization of Gibbs's lemma.

Let vector  $g = \{g_1, g_2, \dots, g_m\}$ , satisfying the limitations

$$g_i \geq 0 \text{ and } \sum_{i=1}^m g_i = \text{const}, \quad (6.40)$$

it minimize the function

$$\Phi(g) = \sum_{i=1}^m f_i(g_i, K_i),$$

where  $K_i$  - linear form

$$K_i = \sum_{p=1}^m Q_{ip} g_p,$$

moreover function  $f_i(g, K)$  they are differentiated of  $g$  and of  $K$ . Then there are a constant  $\lambda$ , such, that

$$\frac{\partial}{\partial g} f_i(g_i, K_i) + \sum_{p=1}^m Q_{pi} \frac{\partial}{\partial K} f_i(g_i, K_i) \underset{\substack{\geq \lambda \frac{\partial}{\partial g} f_i(g_i, K_i) \\ \geq \lambda \frac{\partial}{\partial K} f_i(g_i, K_i)}}{\overset{f_i}{\geq}} \begin{matrix} g_i > 0, \\ g_i = 0. \end{matrix} \quad (6.41)$$

Key: (1). with.

Let us demonstrate this confirmation. Let  $g_n > 0$ . Let us take similar  $\varepsilon \geq 0$ , that  $g_n - \varepsilon \geq 0$ . Then the vector

$$g' = \{g'_i\} = \begin{cases} g_i & \text{if } i \neq n, j, \\ g_i - \varepsilon & \text{if } i = n, \\ g_i + \varepsilon & \text{if } i = j \end{cases}$$

Key: (1). with

it satisfies limitations (6.40). Therefore, in view of condition,  $\Phi(g) \leq \Phi(g')$ . i.e.

$$\sum_i f_i(g_i, K_i) \leq \sum_i f_i(g_i, K'_i) + f_n(g_n - \varepsilon, K'_n) + f_j(g_j + \varepsilon, K'_j),$$

where

$$K'_i = \sum_p Q_{ip} g'_p = \sum_p Q_{ip} g_p - \varepsilon (Q_{in} - Q_{ij}) = K_i - \varepsilon (Q_{in} - Q_{ij}).$$

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Latter/last inequality can be rewritten in the form

$$\begin{aligned} \sum_i [f_i(g_i, K_i) - f_i(g_i, K_i - \varepsilon (Q_{in} - Q_{ij}))] &< \\ &< f_n(g_n - \varepsilon, K'_n) + f_j(g_j + \varepsilon, K'_j) - f_n(g_n, K'_n) - f_j(g_j, K'_j). \end{aligned}$$

After dividing both parts on  $\varepsilon$  and after passing to limit  $\varepsilon \rightarrow 0$ , after simple conversions we will obtain

$$\begin{aligned} \frac{\partial}{\partial g} f_n(g_n, K_n) + \sum_i Q_{in} \frac{\partial}{\partial K} f_i(g_i, K_i) \leq \\ \leq \frac{\partial}{\partial g} f_j(g_j, K_j) + \sum_i Q_{ij} \frac{\partial}{\partial K} f_i(g_i, K_i). \end{aligned} \quad (6.42)$$

If also  $g_j > 0$ , then on the same foundations correctly reverse/inverse inequality. Consequently, when  $g_n, g_j > 0$  in (6.42) occurs equal sign, i.e., there is a constant  $\lambda$ , which satisfies upper line (6.41). If  $g_j = 0$ , is correct inequality (6.42), which corresponds to lower line (6.41). Confirmation is proved.

When functions  $f_i$  do not depend on  $K_n$  it is obtained Gibbs's usual lemma [90]. The previous proof also, in the main thing, is repeated [90].

It is not difficult to see that occurs also the analogous confirmation in the continuous version.

Let function  $g(\omega)$ , satisfying the limitations

$$g(\omega) \geq 0 \text{ and } \int_a^b g(\omega) d\omega = \text{const.}$$

minimize the functional

$$\Phi(g) = \int_a^b f(\omega, g(\omega), K(\omega)) d\omega,$$

where  $K$  - linear operator -

$$K(\omega) = \int_a^b Q(\omega, \omega') g(\omega') d\omega'.$$

moreover  $f(\omega, g, K)$  it is differentiated on  $g$  and on  $K$ . Then there are a constant  $\lambda$ , such, that

$$\frac{\partial}{\partial g} f(\omega, g, K) + \int_a^b Q(\omega', \omega) \frac{\partial}{\partial K} f(\omega', g, K) d\omega' = \begin{cases} \lambda & \text{if } g(\omega) > 0, \\ 0 & \text{if } g(\omega) = 0. \end{cases} \quad (6.43)$$

Key: (1). with.

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For the proof it suffices to decompose interval  $(a, b)$  in the arbitrarily low sections and to replace integrals with sums, and then, after using (6.41), to return to the integral form.

Specifically, the continuous version of Gibbs's lemma was used in §6.7 for the conclusion/output of condition (6.33). In this case if  $f$  does not depend on  $K$  and on the left side (6.43) there remains

only first component/term/addend.

Let us note also that the proved necessary condition of minimum (6.43) can be interpreted otherwise. As usual, the normalization condition

$$E(g) = \int_a^b g(\omega) d\omega = \text{const}$$

can be taken into account with the help of Lagrange's factor  $\lambda$ , i.e., task consists of the minimization of the new functional

$$\Phi_1(g) = \Phi(g) - \lambda E(g)$$

during limitation  $g \geq 0$ . In this treatment (6.43) has the form

$$\Phi'_1(g) = \begin{cases} 0 & \text{при } g > 0, \\ \geq 0 & \text{при } g = 0. \end{cases}$$

Key: (1). with

where  $\Phi'_1 = \text{grad} \Phi_1$  - derivative of the functional (see §1.9).

This result is natural. Limitation  $g(\omega) \geq 0$  determines many permissible functions, the condition  $g(\omega) = 0$  indicating its "boundary".

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If the minimum of functional occurs at any internal point of set,

then, naturally, in its derivative is equal to zero. But if the minimum (is more precise the lower bound) it is reached on the boundary, derivative at the appropriate point can be positive.

These considerations prompt that the result, close to (6.43), must occur, also, with the more common format of functional  $\Phi$ , but this generalization by us will not be necessary.

#### 6.10. Optimization of signal during the matched processing.

Being returned to minimization (6.38), let us note that in this case

$$\frac{\partial f}{\partial g} = K + 2c\tilde{F} - 2cg, \quad \frac{\partial f}{\partial K} = g.$$

Therefore, applying (6.43) and taking into account the symmetry of kernel  $Q(\omega - \omega')$ , we come to condition (6.44):

$$cg(\omega) - \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega - \omega') g(\omega') d\omega' = c[\tilde{F}(\omega) - \lambda] \text{ при } g(\omega) > 0, \\ \leq c[\tilde{F}(\omega) - \lambda] \text{ при } g(\omega) = 0.$$

Key: (1). with

moreover the constant  $\lambda$  is determined and the condition for standardization (6.32).

The method of deciding the integral equation, which corresponds to upper line (6.44) is well known. However, the presence of inequality in the lower line does not make it possible to directly use this method, and is required thinner consideration.

Let us assume that the correlation function  $R(t)$ , connected with  $g(\omega)$  with Fourier transform

$$g(\omega) = \int_{-\infty}^{\infty} R(t) e^{-j\omega t} dt, \quad (6.45)$$

decreases rather rapidly with large  $t$ , i.e., belongs to  $L^2$ . With this, we virtually do not reduce generality: such  $R(t)$  are possessed, for example, ~~all~~ all signals of finite energy whose spectrum is contained in an arbitrarily wide finite band. Then we can select such a large time interval  $(-T, T)$ , first having limited the area of integration in (6.45) by it, we obtain an air arbitrarily small distortion  $g(\omega)$ . In other words, taking a sufficiently large  $T$ , we can accept

$$g(\omega) = \int_{-T}^T R(t) e^{-j\omega t} dt.$$

We substitute the last expression in the left side of (6.44) and consider that  $Q(\omega)$  is determined in accordance with (6.24). This provides

$$\begin{aligned} cg(\omega) - \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega - \omega') g(\omega') d\omega' = \\ = \int_{-T}^T R(t) \left( c - \frac{\sin t\Delta}{\pi t} \cos \nu t \right) e^{-j\omega t} dt. \end{aligned}$$

We see that the left side of (6.44) is an analytical function of  $\omega$ . We also assume that the functions  $F(\omega)$ , which stands in the right side is analytical, i.e., for example, that a given  $F(t)$  has finite duration.

Further, assume that in some frequency interval  $g(\omega) > 0$  and, consequently, the equality which corresponds to the upper line of (6.44) is satisfied. Since the functions of both sides of (6.44) are analytical, this equality will be formally satisfied with all  $\omega$  and not only in the indicated interval. This means that where  $g(\omega)$  is positive, it agrees with the solution of the equation

$$g(\omega) - \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega - \omega') g(\omega') d\omega' = \tilde{F}(\omega) - \lambda, \quad (6.46)$$

which is obtained from (6.44) without consideration of the lower line. Using the normal procedure, (see [93], sec. 11.1) we take the Fourier transform from both parts (6.46) and after simple facings we will obtain

$$\begin{aligned}
 g(\omega) &= \int_{-\infty}^{\infty} \frac{F(t) - \lambda \delta(t)}{1 - \frac{\sin t\Delta}{\pi t} \cos \omega t} e^{-i\omega t} dt = \\
 &= \int_{-\infty}^{\infty} \frac{F(t) e^{-i\omega t} dt}{1 - \frac{\sin t\Delta}{\pi t} \cos \omega t} - \frac{\lambda}{1 - \Delta/\pi}, \quad (6.47)
 \end{aligned}$$

where  $F(t)$  the assigned function, see (6.29).

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Let us emphasize again that this formula is accurate, only if obtained  $g(\omega)$  is positive. If right side gives negative value, one should according to lower line (6.44) to place  $g(\omega) = 0$ , i.e., produce the "cutting" of negative values as in §4.6.

Formula (6.47) gives the unique in  $L^2$  solution of equation (6.46) [93]. From other side the operation "cuttings" is also implemented by only form, since the unknown constant  $\lambda$  is uniquely

determined from the condition for the standardization (see below).

Thus, if the solution of our problem exists in  $L^2$ , it is singular.

However, decision exists not at all values of parameter  $c$ . From (6.47) it is evident that with some  $c$  the denominator of integrand can become zero, then integral diverges, and  $g(\omega)$  cannot be determined.

At the arbitrary medium frequency of zone  $V$  the decision exists, only if  $c > \Delta/\nu$  <sup>1</sup>.

FOOTNOTE 1. This limitation of allowed values of  $c$  is substantial during attempts at the numerical solution of task. It is necessary to choose  $c$  from the permissible region, otherwise result can be absurd. But without having analytical decision, this field is virtually determine difficultly; apparently, there is no indications of physical character, concerning the appropriate selection of Lagrange's  $c$  - indefinite factor in functional (6.30). At least, our attempts to fulfill numerically minimization (6.30) proved to be unsuccessful for this reason. This is one example where the numerical methods are barely effective. ENDFOOTNOTE.

Let us consider the important practical case. We will attempt to obtain high the practical case. We will attempt to obtain high resolution in the time. Then desired  $F(t)$  has short duration  $\tau$ , such that  $\tau\Delta \ll 1$  and  $\tau\omega \ll 1$ , i.e. the interference zone is placed in the center section of spectrum  $\tilde{F}(\omega)$ . Then integral (6.47) has final limits, moreover denominator little is changed in the range of integration. Therefore

$$g(\omega) = \frac{\int_{-1}^1 F(t) e^{-i\omega t} dt - \lambda}{1 - \Delta/\pi c} = c_1 \tilde{F}(\omega) - \lambda_1.$$

The construction of the spectrum is shown in Fig. 6.5. After assigning certain  $c_1 > 1$ , we must increase spectrum  $\tilde{F}(\omega)$  in  $c_1$  times, and then displace in the vertical line by  $\lambda_1$  and "to cut" negative values.

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The amount of displacement is selected so as to satisfy standardization (6.32), i.e., so that the area of positive segment of a curve would be equal to unity.

Is the more  $c_1$ , the greater the displacement  $\lambda_1$ , and by the fact in the smaller band is included spectrum  $g(\omega)$ . Choosing sufficiently large  $c_1$ , it is possible to narrow down  $g(\omega)$  so, that the space of

the body of uncertainty/indeterminacy in the interference zone will be equal to zero. But interference zone is arranged/located in the center section of spectrum  $\tilde{F}(\omega)$ , therefore in order to obtain zero space in the zone, it is necessary to take  $g(\omega)$ , therefore in order to obtain zero space in the zone, it is necessary to take  $g(\omega)$  substantially narrower than  $\tilde{F}(\omega)$ . The quality of the approximation/approach of correlation function to assigned  $F(t)$ , naturally, will be in this case poor. On the other hand, expanding  $g(\omega)$ , we will obtain better approximation/approach to  $F(t)$ , but space in the interference zone sharply will increase. We see that the conditions of low partial space in the band and good approximations/approaches to a narrow single-peak correlation function substantially contradict each other. During the stringent requirements for the value of partial space to find a satisfactory practical compromise hardly possible.

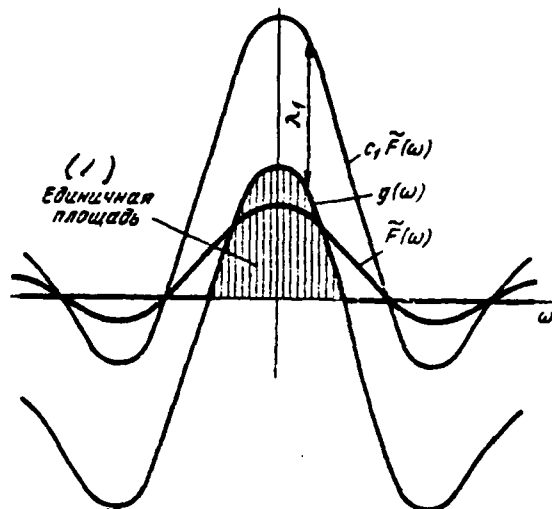


Fig. 6.5.

Key: (1). Single area.

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Let us note also that case  $c_1=1$  leads to the task about the best approximation of autocorrelation function to assigned unrealizable  $F(t)$  (cf §4.6). From the results of chapter 4 it is clear that at least in many instances, we will obtain the single-peak correlation functions, similar to that shown in Fig. 4.3. This result contradicts, for example, the assumptions of work [92].

However, during another location of interference zone are

obtained multipeak correlation functions. In this connection let us consider another example. Let the sufficiently narrow interference zone be arranged/located on certain removal/distance from the center section of spectrum  $\tilde{F}(\omega)$ . Then (6.47) accepts the form

$$g(\omega) = \int_{-\tau}^{\tau} \frac{F(t)}{1 - \frac{\Delta}{\pi c} \cos vt} e^{-i\omega t} dt - \lambda_1.$$

After using further next by Fourier

$$\frac{1}{1 - \alpha \cos vt} = \frac{1}{V1 - \alpha^2} \sum_{n=-\infty}^{\infty} \left( \frac{1 - V1 - \alpha^2}{\alpha} \right)^{|n|} e^{invt},$$

we obtain

$$g(\omega) = \frac{1}{V1 - \alpha^2} \sum_{n=-\infty}^{\infty} \left( \frac{1 - V1 - \alpha^2}{\alpha} \right)^{|n|} \tilde{F}(\omega - n\nu) - \lambda_1 > 0;$$

$$\alpha = \frac{\Delta}{\pi c} < 1.$$

It is clear that  $g(\omega)$  is constructed similarly to previous, but instead of the single spectrum  $\tilde{F}(\omega)$  is used the sum of such spectra, frequency-displaced on  $\pm\nu, \pm 2\nu$ , and so forth and undertaken with the decreasing coefficients. If  $\nu$  is sufficiently great, these spectra are considerably spread, and afterward cuttings will remain not one, but several lobes/lugs of the spectrum. As a result of  $g(\omega)$  it approaches the periodic structure, shown in Fig. 6.4, and  $R(t)$  becomes multipeak.

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Chapter 7.

APPROXIMATION OF THE REALIZABLE FUNCTIONS OF INDETERMINANCY AND  
AUTOCORRELATION FUNCTIONS.

Examining in the previous chapters the synthesis of the functions of uncertainty/indeterminacy and autocorrelation functions, we did not assign any limitations on the permissible signals. With this approach are revealed/detected the maximum possibilities of approximation/approach, at best are determined the optimum signals, which realize these possibilities, but all this it is done without taking into account that, how technically are difficult to achieve these or other signals. In particular, the signals, examined in chapter 6, maximize the partial space of the body of uncertainty/indeterminacy, but they are complicated for the realization, since is required a deep amplitude modulation at the high power level.

Therefore, beginning the synthesis, it is expedient to bound many permissible signals by knowingly realizable in the predicted equipment. In particular for the radar of main interest are signals

with the frequency modulation or the phase manipulation with rectangular envelope.

A similar limitation of many permissible signals narrows the possibilities of selection, and the quality of approximation/approach frequently proves to be considerably worse, than with the signals of arbitrary form. Therefore it is possible to consider "ideal" the results, obtained with the arbitrary signals and, without examining more general problem, to be bounded to approximation/approach to the realizable function of uncertainty/indeterminacy or autocorrelation function, optimum on many all physically realizable signals.

It is obvious, we come to the fundamental task of synthesis in the space of signals. The realizable function of uncertainty/indeterminacy (autocorrelation function), approximation/approach to which is found, determines many desired signals  $Y$ . Each signal  $y \in Y$  possesses similar functions of indefiniteness or autocorrelation function. Finding out the minimum of the distance between  $Y$  and many permissible signals  $X$ , it is possible to determine the signal, nearest to the desired set.

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In accordance with the hypothesis of proximity this permissible:

signal provides best approximation to the desired property, i.e., to the assigned realizable function of uncertainty/indeterminacy or autocorrelation function.

However, it is important to establish, as approximations/approaches in the space of signals they are connected with the approximations/approaches of the functions of uncertainty/indeterminacy and autocorrelation functions, what condition of optimum character satisfy the latter, if signals satisfy the criterion of proximity. In other words, it is necessary to explain, is applicable the hypothesis of proximity to the tasks in question. Two theorems of present chapter positively answer this question.

#### 7.1. Synthesis according to the function of uncertainty/indeterminacy.

Let there be the function of uncertainty/indeterminacy  $\gamma_s(t, \Omega)$ , realized by certain signal  $s(t)$ . Let us explain first of all, is the realizing signal only, are there other signals and what they must be so that the function of uncertainty/indeterminacy would coincide with given one  $\gamma_s(t, \Omega)$ .

Let us turn to the determination of the function of

uncertainty/indeterminacy (5.1). This relationship/ratio can be, obviously, considered as Fourier transform on variable  $t'$ . Therefore on the basis of inverse transformation of Fourier we have

$$s\left(t' + \frac{t}{2}\right)s^*\left(t' - \frac{t}{2}\right) = \frac{E}{2\pi} \int_{-\infty}^{\infty} \chi_s(t, \Omega) e^{-j\Omega t'} d\Omega.$$

Or assuming/setting

$$t' + \frac{t}{2} = u, \quad t' - \frac{t}{2} = v,$$

$$s(u)s^*(v) = \frac{E}{2\pi} \int_{-\infty}^{\infty} \chi_s(u-v, \Omega) \exp\left(-j\frac{u+v}{2}\Omega\right) d\Omega. \quad (7.1)$$

This relationship/ratio is the condition of the feasibility of the function of the uncertainty/indeterminacy: function  $\chi_s(t, \Omega)$  is realized as the function of uncertainty/indeterminacy in that and only in such a case, when integral is to the right the product of two identical compositely harressed factors<sup>1</sup>.

FOOTNOTE <sup>1</sup>. From previous follows only the need for condition (7.1): sufficiency it is easy to show, implementing the reverse/inverse replacement of variable/alternating and Fourier transform. This brings to (5.1). ENDFOOTNOTE.

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Further, after assuming here  $v=0$  and again changing designations, we find

$$s(t)s^*(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi_s(t, \Omega) e^{-j\frac{\Omega t}{2}} d\Omega. \quad (7.2)$$

Here we did not write out value  $E$ , after taking the normalization condition

$$E = \int |s(t)|^2 dt = 1. \quad (7.3)$$

From (7.2) it is clear that the function of uncertainty/indeterminacy  $\chi_s(t, \Omega)$  determines the realizing signal  $s(t)$  with an accuracy to the factor

$$s^*(0) = |s(0)| e^{-j \arg s(0)} = |s(0)| e^{j \phi_0}.$$

Amplitude  $|s(0)|$  is uniquely determined further by the condition for standardization (7.3), but initial phase  $\phi_0$  remains arbitrary. Consequently, two signals, realizing one and the same function uncertainties/indeterminacies, can be characterized by only initial phase. It is not difficult to note that this condition is also sufficient: if signals are characterized by only initial phase, then they have the identical functions of uncertainty/indeterminacy.

Actually/really, product  $s[t' + (t/2)]s^*[t' - (t/2)]$ , obviously, does not depend on initial phase, but the function of uncertainty/indeterminacy (5.1) contains signal only in the form of the product indicated.

Thus, let there be the function of uncertainty/indeterminacy

$\chi_s(t, \Omega)$ , realized by signal  $s(t)$ . According to proved the same function of uncertainty/indeterminacy have all signals of the form

$$y(t) = s(t) e^{j\omega_0 t}, \quad (7.4)$$

and any signal, different from (7.4), it has another function of uncertainty/indeterminacy. Therefore, examining the task of synthesis according to the realizable function of uncertainty/indeterminacy  $\chi_s(t, \Omega)$ , we they must include/connect in the desired set  $Y$  the signals of form (7.4), which differ only in terms of initial phase from each other.

Further, let there be an arbitrary multitude of the permissible signals  $X$ .

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Let us explain, what condition of optimum character satisfies signal  $x_{opt}$  nearest to set  $Y$ . For this let us fix first certain permissible signal  $x \in X$  let us find shortest distance from this signal to set  $Y$ . As usual, it is necessary to determine the coefficient of the proximity

$$C(x, Y) = \max_{y \in Y} C(x, y) = \max_{y \in Y} \operatorname{Re} \int_{-\infty}^{\infty} x(t) y^*(t) dt.$$

Taking into account (7.4) we have further:

$$C(x, Y) = \max_{\varphi_0} \operatorname{Re} e^{-j\varphi_0} \int_{-\infty}^{\infty} x(t) s^*(t) dt = \\ = \max_{\varphi_0} \operatorname{Re} e^{-j\varphi_0} (x, s).$$

It is here taken into consideration, that the signals of set  $Y$  are characterized by only initial phase; therefore maximization is produced on  $\varphi_0$ . Value  $(x, s)$  designates scalar product. After using the identity

$$(x, s) = |(x, s)| e^{j \arg(x, s)},$$

we have further

$$C(x, Y) = |(x, s)| \max_{\varphi_0} \cos[\arg(x, s) - \varphi_0].$$

Maximum on  $\varphi_0$ , obviously, reaches at  $\varphi_0 = \arg(x, s)$ ; therefore

$$C(x, Y) = |(x, s)|.$$

In order to obtain shortest distance  $d_{min}$  between sets  $X$  and  $Y$ , it is necessary to maximize the coefficient of proximity also in signals  $x(t)$ , i.e.,

$$C(X, Y) = \max_{x \in X} C(x, Y) = \max_{x \in X} |(x, s)|,$$

and

$$d_{min}^2 = 2[1 - C(X, Y)] = 2[1 - \max_{x \in X} |(x, s)|]. \quad (7.5)$$

Thus, the shortest distance between sets  $X$  and  $Y$  realizes signal  $x_{opt}$ , which maximizes the modulus/module of scalar product  $(x, s)$ .

Let us now show that the same condition satisfies the signal which minimizes a quadratic difference in the functions of uncertainty/indeterminacy, i.e.,

$$d^2(\gamma_s, \gamma_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\gamma_s(t, \Omega) - \gamma_x(t, \Omega)|^2 dt d\Omega = \min. \quad (7.6)$$

Actually/really, taking into account condition (5.4), it is not difficult to obtain

$$d^2(\gamma_s, \gamma_x) = 2[1 - C(\gamma_s, \gamma_x)], \quad (7.7)$$

where

$$C(\gamma_s, \gamma_x) = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma_s(t, \Omega) \gamma_x^*(t, \Omega) dt d\Omega$$

- coefficient of the proximity of the functions of uncertainty/indeterminacy. For calculating the latter we will use the conversion of Sussmar (5.20). This it gives

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma_s(t, \Omega) \gamma_x^*(t, \Omega) dt d\Omega &= \\ &= \gamma_{xs}(0, 0) \gamma_{xs}^*(0, 0) = |\gamma_{xs}(0, 0)|^2 = \\ &= \left| \int_{-\infty}^{\infty} x(t) s^*(t) dt \right|^2 = |(x, s)|^2. \end{aligned} \quad (7.8)$$

Is here taken into consideration also the determination of the cross function of uncertainty/indeterminacy (5.19). Thus,

$$d^2(\gamma_s, \gamma_x) = 2[1 - |(x, s)|^2] = 2[1 - C^2(x, Y)]. \quad (7.9)$$

Comparison (7.5) and (7.9) leads to the following theorem:

In order to obtain the best quadratic approximation of the functions of uncertainty/indeterminacy, it suffices to find the signal, which realizes the shortest distance between sets  $X$  and  $Y$  in the space of signals  $H$ .

We obtained the convincing confirmation of the hypothesis of proximity. In the task in question quadratic approximations/approaches in the space of signals proved to be completely equivalent to the same approximations/approaches in function space of uncertainty/indeterminacy.

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Let us note that a special case of this theorem is proved by Sussman. In [72] it is shown that if set  $X$  is a linear variety of a finite number of measurements (hyperplane), then the approximation/approach of the functions of uncertainty/indeterminacy  $\chi_s$  and  $\chi_x$  (in the sense of criterion (7.6)) it is reduced to the design of signal  $s(t)$  to set  $X$ . It is obvious, under given conditions this is equivalent to the minimization of the distance between  $s$  and  $X$ .

FOOTNOTE 1. Phase  $\phi_0$  in this case role does not play, since set  $X$  contains signals with the arbitrary initial phases. A change in the phase of signal  $y$  leads only to a change in the phase of signal  $x_{opt}$  without affecting the value of distance. ENDFOOTNOTE.

Our proof is applicable to any set  $X$ . In this form the theorem can be used, in particular, with the synthesis of signals with the frequency modulation or the phase manipulation.

## 7.2. Synthesis according to the autocorrelation function.

Let us consider the analogous task when it is necessary to find approximation/approach to the realizable autocorrelation function, but not to the function of uncertainty/indeterminacy.

In this case set  $Y$  must include all signals, which possess the assigned autocorrelation function. But autocorrelation function uniquely determines the spectrum of the power of the signal

$$a^2(\omega) = |\tilde{s}(\omega)|^2 = \int_{-\infty}^{\infty} R(t) e^{-j\omega t} dt, \quad (7.10)$$

therefore signals  $y(t) \in Y$  have one and the same amplitude spectrum  $a(\omega)$ , depending on assigned  $R(t)$ :

$$\tilde{y}(\omega) = a(\omega) e^{-j\alpha(\omega)}. \quad (7.11)$$

Phase spectrum  $\alpha(\omega)$  is arbitrary, in terms of this differs one signal

of set Y from another.

Let there be the arbitrary permissible signal  $x(t)$  with the spectrum

$$\tilde{x}(\omega) = b_x(\omega) e^{-j\beta_x(\omega)}, \quad (7.12)$$

As with the synthesis according to the function of uncertainty/indeterminacy, let us determine the first shortest distance between signal  $x$  and set Y.

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Let us show the following theorem:

a). Best approximation to signal  $x(t)$  with spectrum (7.12) gives on set Y signal  $y(t)$  whose spectrum is determined by the condition

$$\tilde{y}(\omega) = a(\omega) e^{-j\beta_x(\omega)} \quad (7.13)$$

for all values  $\omega$ , at which  $b_x(\omega) \neq 0$ .

b). If amplitude spectrum  $b_x(\omega)$  is different from zero in any interval  $\omega$  of final measure, signal of best approximation/approach on set Y only.

c). Minimum distance between signal  $x(t)$  and set Y and corresponding coefficient of proximity comprise

$$d^2(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [a(\omega) - b_x(\omega)]^2 d\omega. \quad (7.14)$$

$$C(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b_x(\omega) d\omega. \quad (7.15)$$

For proof we will use the representation of the coefficient of proximity through the spectra of signals, see (1.21),

$$C(x, y) = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) \tilde{y}^*(\omega) d\omega.$$

Substituting the values of  $\tilde{x}(\omega)$  and  $\tilde{y}(\omega)$  from (7.11) and (7.12), we get

$$C(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b_x(\omega) \cos[\alpha(\omega) - \beta_x(\omega)] d\omega. \quad (7.16)$$

According to theorem conditions the amplitude spectra  $a(\omega) \geq 0$  and  $b_x(\omega) \geq 0$  are here assigned. Is assigned also phase spectrum  $\beta_x(\omega)$  of signal  $x(t)$ . We should maximize the coefficient of proximity  $C(x, y)$ , selecting signal  $y(t)$ , i.e., varying phase spectrum  $\alpha(\omega)$ .

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But, as it is clear from (7.16),

$$C(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b_x(\omega) d\omega$$

equality is reached in that and only in such a case, when  $\alpha(\omega) = \beta_x(\omega)$ .

for all values  $\omega$ , at which  $a(\omega) = 0$   $b_x(\omega) \neq 0$  <sup>1</sup>.

FOOTNOTE <sup>1</sup>. With exception of an arbitrary multitude of zero measure. The signals, which differ on the null set, in space  $H$  are not distinguished. Such signals have identical autocorrelation functions, functions of uncertainty/indeterminacy and, moreover, give identical output potential of any realizable receiver. ENDFOOTNOTE.

Since with  $a(\omega) = 0$  the phase spectrum of signal  $y(t)$  is not determined, this serves as proof to the first two confirmations of theorem. The third confirmation directly follows from formulas (7.12), (7.13) and from the determinations of distance and coefficient of proximity (1.7) and (1.21). Theorem is proved.

Let us emphasize, that for the uniqueness of the best approximation it is significant that amplitude spectrum  $b_x(\omega)$  is different from zero in any finite frequency range. If for certain interval  $(\omega_1, \omega_2)$  spectrum  $b_x(\omega) = 0$ , then, as it follows from (7.16), phase spectrum  $\alpha(\omega)$  can be arbitrary in this interval: the value of the coefficient of proximity (but, therefore, and distance) does not depend on  $\alpha(\omega)$ . Thus, best approximation is ambiguous.

The formulated theorem it is not difficult to interpret. Since the signals, which belong to set  $Y$ , have arbitrary phase spectra

$\alpha(\omega)$ , and the amplitude spectrum  $a(\omega)$  is fixed/recorded, finding out approximation/approach to certain signal  $x(t)$ , logical to ascribe  $a(\omega)$  the phase spectrum of signal  $x(t)$ . the differences between the signals, the distance between them will then depend only on the unvariable amplitude spectra, that also is expressed by formulas (7.14), (7.15).

This theorem will be further used with the synthesis of ChM and FM signals. We will now obtain with its aid the condition of optimum character which satisfies signal  $x_{opt}$ , nearest to set  $Y$ .

In order to obtain shortest distance  $d_{min}$ , it is necessary to minimize right side (7.14) also on signals  $x(t)$ , i.e.,

$$d_{min}^2 = \min_{x \in X} d^2(x, Y) = \min_{x \in X} \frac{1}{2\pi} \int_{-\infty}^{\infty} [a(\omega) - b_x(\omega)]^2 d\omega \quad (7.17)$$

or, which is equivalent,

$$C(X, Y) = \max_{x \in X} \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b_x(\omega) d\omega. \quad (7.18)$$

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Thus, the optimum permissible signal  $x \in X$ , realizing distance  $d_{min}$ , gives the best quadratic approximation of amplitude spectrum  $b_x(\omega)$  to the assigned amplitude spectrum  $a(\omega)$ .

As it was noted, the amplitude spectrum of signal is mutually unambiguously connected with its correlation function. Therefore the approximation/approach of the amplitude spectra, attained at the use/application of a criterion of proximity, provides the specific approximation/approach of the correlation function of signal to the given one. In particular, from (7.10) follows the identity

$$\int_{-\infty}^{\infty} |R(t) - R_x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} [a^2(\omega) - b_x^2(\omega)]^2 d\omega. \quad (7.19)$$

showing that the best quadratic approximation of correlation functions is achieved by the analogous approximation/approach of the spectra of power - squares of the amplitude spectra.

Moreover, the value standard deviation of correlation functions can be approximately connected with the distance between the signals. For this let us do some completely acceptable assumptions.

Let us assume that signals  $x(t) \in X$  have the final duration  $T$ . This corresponds, in particular, to examined/considered further FM and ChM signals. Then, autocorrelation function  $R_x(t)$  is different from zero in the interval  $(-T, T)$ . Assuming also that this interval contains the most essential part of the assigned function  $R(t)$ , let us determine root-mean-square error  $\delta$  by the relationship/ratio

$$\delta^2 = \frac{1}{2T} \int_{-\infty}^{\infty} |R(t) - R_x(t)|^2 dt. \quad (7.20)$$

Taking into account (7.19) we have further

$$\delta^2 = \frac{1}{4\pi T} \int_{-\infty}^{\infty} [a(\omega) + b_x(\omega)]^2 [a(\omega) - b_x(\omega)]^2 d\omega. \quad (7.21)$$

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Using the law of mean, let us take out the first factor for the integral, after taking it at certain midpoint of the axis of frequencies. Taking into account (7.14), we obtain

$$\delta^2 = \frac{1}{2T} [a(\omega) + b_x(\omega)]^2 d^2(x, Y). \quad (7.22)$$

After preserving acceptable for our evaluation/estimate accuracy, it is possible to considerably simplify this relationship/ratio. First, we will count the approximation/approach of the spectra sufficiently to good ones, so that for the interesting us medium frequency

$$b_x(\omega) \approx a(\omega) \approx a_{cp}(\omega).$$

In the second place, let us determine the effective bandwidth  $\Omega$ , occupied by the assigned spectrum  $a(\omega)$ , being based on the standardization on the energy:

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} a^2(\omega) d\omega \approx \frac{1}{2\pi} a_{cp}^2(\omega) \cdot 2\Omega.$$

Therefore  $a_{cp}^2(\omega) = \pi \Omega$ .

As a result formula (7.22) takes the form

$$\delta^2 = \frac{2\pi}{\Omega T} d^2(x, Y) = \frac{2}{m} d^2(x, Y), \quad (7.23)$$

where

$$m = \Omega T / \pi \quad (7.24)$$

- contraction coefficient of signal  $x(t)$ . This value is defined as the product of the duration of signal  $T$  to the effective bandwidth of the desired signal  $y(t)$ .

For optimum signal  $x_{opt}$  of that realizing shortest distance  $d_{min}$  from (7.23) we obtain respectively

$$\delta_{min} \approx \sqrt{2/m} d_{min}. \quad (7.25)$$

This result has basic value.

First, establishing the direct dependence between the approximations/approaches in the space of signals and the standard deviation of autocorrelation functions, we confirm the applicability of the hypothesis of proximity to the task in question.

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In the second place, formula (7.25) gives the direct method of the evaluation/estimate of the minimum divergence of autocorrelation functions of distance  $d_{min}$ . For the signals of some types, in

particular for phase-keyed, this makes it possible to come to light/detect/expose the important laws, which characterize the maximum possibilities of the approximation/approach (see Chapter 9).

One should emphasize that the obtained evaluation/estimate is approximate. The minimization of the distance between  $X$  and  $Y$ , providing the best approximation of the amplitude spectra, nevertheless does not guarantee the best approximation of correlation functions. Said relates as to quadratic criterion (7.20)), so, and with the even large foundation, to the minimax criterion, frequently utilized with the synthesis of signals. Therefore, the solution of this task, obtained on the base of the criterion of proximity, front is only initial approximation/approach, and it must be further made more precise with the help of the iterative minimizations. This method we will apply with the synthesis of ChM and FM signals.

### 7.3. A change in the space metrics.

The results of the previous paragraph leave certain dissatisfaction because the approximations/approaches in the space of signals proved to be completely not equivalent to approximations/approaches in the space of autocorrelation functions. In fact we have two different criteria of optimum characters (7.17) and (7.20), which weakly differ from each other. But would be to

preferably previously assign the criterion of approximation/approach for the autocorrelation functions, for example, the minimum of error (7.20), and seek the decision, which strictly corresponds to this criterion.

As it was noted in Chapter 1, it is possible to change the space metrics of signals so that the minimization of the distance between sets X and Y would provide best approximation in the sense of the assigned criterion.

In the case in question it is easy to indicate such metric (it is more precise, quasi-metric). Let the distance between signals  $s_1(t)$  and  $s_2(t)$  be determined in the form

$$d^2(s_1, s_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{s}_1(\omega) - \tilde{s}_2(\omega)|^2 d\omega. \quad (7.26)$$

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After assigning, as earlier, many permissible signals  $X$  (arbitrary) and many desired signals  $Y$ , which possess the assigned autocorrelation function  $R(t)$ , we use a criterion of proximity for finding the optimum signal  $x_{opt}$ .

If we fix certain signal  $x \in X$ , then, after repeating the considerations of previous section, it is possible to show that short distance to set  $Y$  comprises

$$d^2(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [a^2(\omega) - b_x^2(\omega)]^2 d\omega, \quad (7.27)$$

the nearest to  $x$  signal  $y \in Y$  having the same phase spectrum, i.e.,  $\alpha(\omega) = \beta_x(\omega)$ .

Comparison (7.27) and (7.19) shows that minimization of the distance between  $X$  and  $Y$  in the space in question is equivalent to the best quadratic approximation of autocorrelation functions. Thus, after selecting special metric, we actually/really arrived at the complete agreement with the assigned criterion. However, metric (7.26) we will not in practice use for the synthesis of signals. This

is connected with the fact that with this metric the condition of single energy of signal does not coincide with the condition of single norm. The corresponding complications, generally speaking, are surmounted, for example, with the help of the simplex method (see Chapter 4, where we met with a similar difficulty), but for this task are preferable other iterative methods.

#### 7.4. Special features/peculiarities of the synthesis of composite/compound signals.

Many signals, which have practical use/application in the radar, are coherent bursts of pulses - words, which consist of the repeating, elementary impulses/pulses (discretes) of the assigned form. Such signals we will call composite/compound. They include, in particular, the quantified FM signals, in detail examined/considered further.

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In general composite/compound signal can be registered in the form

$$x(t) = \sum_{i=1}^N x_i u_i(t - t_i) \quad (7.28)$$

where  $x_i$  - composite amplitudes of samples, and  $t_i$  - their displacement in the time. As a rule, separate samples do not overlap.

In order conveniently to reflect this special feature/peculiarity, we will assume the following.

The duration of the single elementary signal  $u_0(t)$  let us place single:

$$u_0(t) \begin{cases} \neq 0 \text{ при } -1/2 \leq t \leq 1/2; \\ = 0 \text{ при } |t| > 1/2 \end{cases}$$

Key: (1). with. and we normalize, furthermore, its energy:

$$E_0' = \|u_0\|^2 = \int_{-1/2}^{1/2} |u_0(t)|^2 dt = 1. \quad (7.29)$$

Moments/torques  $M_k$  which characterize the order of elementary samples in the time, we will assure/set by whole numbers. By this is excluded, obviously, the overlap of samples in the time. Finally, we normalize also energy of composite/compound signal as a whole. Taking into account what has been said we come to the condition

$$\|x\|^2 = \sum_{i=1}^n \|x_i\|^2 = 1. \quad (7.30)$$

In accordance with (7.28) the spectrum of composite/compound signal has the expression

$$\tilde{x}(\omega) = \tilde{u}_0(\omega) \sum_{i=1}^n x_i e^{-j\omega t_i} = \tilde{u}_0(\omega) H(\omega), \quad (7.31)$$

where

$$\tilde{u}_0(\omega) = \int_{-1/2}^{1/2} u_0(t) e^{-j\omega t} dt \quad (7.32)$$

- spectrum of elementary impulse/momentum/pulse.

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Value

$$H(\omega) = \sum_{l=1}^n x_l e^{-j\omega t_l} \quad (7.33)$$

we will call the spectrum of the code. Since  $t_l$  - whole numbers,  $H(\omega)$  - the periodic function:

$$H(\omega) = H(\omega + 2\pi).$$

Let us note also that in accordance with (7.32)  $\tilde{u}_0(\omega)$  is a Fourier transform from the function, finite in interval  $(-1/2, 1/2)$ . Therefore  $\tilde{u}_0(\omega)$  is the integral function of degree of  $1/2$  [83]. Some properties of the integral functions of first degree are used below.

The synthesis of composite/compound signal is reduced to the rational selection of composite amplitudes  $x_l$  and order of impulses/momenta/pulses in the time, characterized by values  $t_l$ . In this case they strive, mainly, not to distort the autocorrelation function of single sample, which, in particular, indicates the low level of the remainders/residues of the obtained correlation function. Examining this task of synthesis, we will consider that the

desired set  $Y$  includes the signals, which possess the correlation function of single the sample

$$R_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{u}_s(\omega)|^2 e^{j\omega t} d\omega,$$

i.e., signals with the amplitude spectrum

$$|\tilde{y}(\omega)| = a(\omega) = |\tilde{u}_s(\omega)|.$$

The permissible set  $X$  includes composite/compound signals (7.28), which satisfy the enumerated above conditions.

applying in this case the hypothesis of proximity taking into account (7.31), in the complete agreement with the theorem of §7.2 we come to the minimization of the value (see (7.14))

$$d^2(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{u}_s(\omega)|^2 [1 - |H(\omega)|]^2 d\omega, \quad (7.34)$$

moreover varied are here parameters  $x_i$  and  $t_i$ , which are determining the spectrum of code  $H(\omega)$ .

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It will be shown below that the task in question allows/assumes equivalent formulation in the space, elements/cells of which are complex amplitudes  $x_i$ . This formulation substantially simplifies the synthesis of composite/compound signals.

Let us preliminarily establish one useful property of the integrals, which contain integral functions [10, 11].

Let  $f(\omega) = f(\omega + 2\pi)$  - periodic function allowing resolution into evenly convergent Fourier series

$$f(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\omega}, \quad (7.35)$$

and  $g(\omega)$  - the whole analytic function of final degree  $\sigma$ , which satisfies one of the following conditions:

a) or  $\sigma < 1$ ,

b) or  $\sigma = 1$  and  $|g(\omega)|$  decreases with  $\omega \rightarrow +\infty$  more rapid than  $1/|\omega|$ .

Then is correct the identity

$$\int_{-\infty}^{\infty} g(\omega) f(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) d\omega \int_{-\infty}^{\infty} f(\omega) d\omega. \quad (7.36)$$

For the proof of this identity let us substitute series/row (7.35) into left side (7.36) and will integrate piecemeal:

$$\int_{-\infty}^{\infty} g(\omega) f(\omega) d\omega = \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} g(\omega) e^{-ik\omega} d\omega = \sum_{k=-\infty}^{\infty} c_k \tilde{g}(k). \quad (7.37)$$

Value

$$\tilde{g}(k) = \int_{-\infty}^{\infty} g(\omega) e^{-i\omega k} d\omega$$

is a Fourier transform from integral function  $g(\omega)$ . In view of the theorem of Wiener-Paley [1, 83] function  $\tilde{g}(\tau)$  is finite in the interval  $(-\sigma, \sigma)$ , i.e.,  $\tilde{g}(\tau) \equiv 0$  when  $|\tau| > \sigma$ . The second of conditions (b) indicates, besides the fact that  $\tilde{g}(\tau)$  is continuous.

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Therefore with satisfaction of conditions a) or b)

$$\tilde{g}^{(k)} = 0 \text{ with } k = \pm 1, \pm 2, \dots$$

As a result in series/row (7.37) there remains only the component/term/addend with  $k=0$ , which, as can easily be seen, and corresponds to right side (7.36). Identity (7.36) is proved.

We convert with the help of this identity integral (7.34), which is determining the distance between the composite/compound signal  $x(\tau)$  and the desired set  $Y$ . For this let us note, in the first place, that function  $f(\omega) = [1 - |H(\omega)|^2]$  has a period  $2\pi$ , since this is correct for  $H(\omega)$ . In the second place, as already mentioned, spectrum  $\tilde{u}_0(\omega)$  there is the integral function of degree of  $1/2$ . With multiplication of integral functions the degree of product does not exceed the sum of the degrees of factors. Therefore the function

$$g(\omega) = |\tilde{u}_0(\omega)|^2 = \tilde{u}_0(\omega) \tilde{u}_0^*(\omega)$$

has a degree not more than 1. This function, furthermore, sufficiently rapidly it decreases with  $\omega \rightarrow +\infty$ , which follows from the limitedness of energy (7.29):

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{u}_s(\omega)|^2 d\omega = 1.$$

Thus,  $g(\omega)$  satisfies the conditions of the previous theorem.

Applying (7.36) to (7.34), we find

$$d^2(x, Y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - |H(\omega)||^2 d\omega. \quad (7.38)$$

In the complete agreement with the criterion of proximity this value should be minimized, selecting the permissible compound signals  $x(t)$ , i.e., varying  $x_i$  and  $t_i$ . Consequently, the task of the synthesis of composite/compound signal in question is reduced to finding  $x_i$  and  $t_i$  with which the spectrum of code  $H(\omega)$  least deviates on the modulus/module from unity.

We will examine further space  $L^2(-\pi, \pi)$  the functions of frequency (spectra), assigned in the interval  $(-\pi, \pi)$ .

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The distance between two such spectra  $x(\omega)$  and  $y(\omega)$  is determined in the form

$$d^2(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{x}(\omega) - \tilde{y}(\omega)|^2 d\omega.$$

Let us show that the task of synthesis (7.38) corresponds to the use/application of a criterion of proximity in this space. Let the permissible set  $X$  turn off/disconnect the spectra of code (7.33)

$$\tilde{x}(\omega) = H(\omega),$$

and the desired set  $Y$  - spectra of single amplitude with arbitrary phases  $\tilde{y}(\omega) = e^{-j\alpha(\omega)}$ . Then

$$d^2(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{-j\alpha(\omega)} - H(\omega)|^2 d\omega. \quad (7.39)$$

The minimization of distance of  $d(x, y)$  corresponds, as usual, to the maximization of the coefficient of the proximity

$$C(x, y) = \operatorname{Re}(x, y) = \operatorname{Re} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\alpha(\omega)} H(\omega) d\omega. \quad (7.40)$$

this maximization can be produced in any order. In particular, fixing/recording certain allowed spectrum  $\tilde{x}(\omega) = H(\omega)$ , we will maximize value  $C$ , selecting phase spectrum  $\alpha(\omega)$ . Maximum reaches at

$$\alpha(\omega) = -\arg H(\omega). \quad (7.41)$$

Substituting this value in (7.39), we come to the shortest distance between the selected with  $x$  and desired set  $Y$

$$d^2(x, Y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{j\arg H(\omega)} - H(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - |H(\omega)||^2 d\omega. \quad (7.42)$$

which must be further minimized on all permissible spectra of code  $d(\omega)$ . The same result will be obtained, if we maximize the

coefficient of the proximity

$$C(x, Y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)| d\omega. \quad (7.43)$$

Relationship/ratio (7.42) coincides with (7.38).

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This proves the admissibility of the use/application of a criterion of proximity (in the version in question) for the solution of our problem of synthesis.

Let us give one additional useful representation for the coefficient of proximity (7.40). Taking into account (7.33), we find

$$C(x, y) = \operatorname{Re} \sum_{i=1}^n x_i \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jx(\omega)} e^{-jy\omega} d\omega.$$

Let us designate through  $y(k)$  Fourier coefficients function  $e^{-jx(\omega)}$ , assigned in the interval  $(-\pi, \pi)$ :

$$y(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jx(\omega)} e^{jk\omega} d\omega, k = 0, \pm 1, \pm 2, \dots$$

According to the condition, values  $t_i$  are whole numbers; therefore

$$C(x, y) = \operatorname{Re} \sum_{i=1}^n x_i y^*(t_i). \quad (7.44)$$

Applying the criterion of proximity, this value should be maximized on all  $x$ , and  $t_i$  that characterize composite/compound signal, and

also on the arbitrary phase spectra  $\alpha(\omega)$  the desired signal.

The task of the synthesis of composite/compound signals in question is reduced, thus, to the use/application of a criterion of proximity in the finite-dimensional (Euclidian) space. The elements/cells of this space are in general the Fourier coefficients of the corresponding spectra. For the composite/compound signal here we have in mind the spectrum of code (7.33), Fourier coefficients which are simply the amplitudes of samples.

As one additional example for the use/application of identity (7.36) we will obtain important representation for functioning the uncertainty/indeterminacy of the composite/compound signal through the spectrum of code [13].

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In accordance with (7.31) we have

$$\begin{aligned} \chi(t, \Omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}\left(\omega - \frac{\Omega}{2}\right) \tilde{x}^*\left(\omega + \frac{\Omega}{2}\right) e^{j\omega t} d\omega = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}_0\left(\omega - \frac{\Omega}{2}\right) \tilde{u}_0^*\left(\omega + \frac{\Omega}{2}\right) H\left(\omega - \frac{\Omega}{2}\right) \times \\ &\quad \times H^*\left(\omega + \frac{\Omega}{2}\right) e^{j\omega t} d\omega. \end{aligned}$$

Functions  $\tilde{u}_0(\omega - \Omega/2)$  and  $\tilde{u}_0^*(\omega + \Omega/2)$  - whole degrees of  $1/2$ , and

$H(\omega - \Omega/2)$  and  $H(\omega + \Omega/2)$  - periodic with the period  $2\pi$ . We will be interested in the values of the function of uncertainty/indeterminacy with the wholes  $t=0, \pm 1, \pm 2, \dots$ . Then  $e^{j\omega t}$  also has period  $2\pi$ , and according to (7.36) is obtained

$$\begin{aligned} \chi(t, \Omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}_0\left(\omega - \frac{\Omega}{2}\right) \tilde{u}_0^*\left(\omega + \frac{\Omega}{2}\right) d\omega \times \\ &\times \frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(\omega - \frac{\Omega}{2}\right) H^*\left(\omega + \frac{\Omega}{2}\right) e^{j\omega t} d\omega = \\ &= \frac{1}{2\pi} \chi_0(t, \Omega) \int_{-\pi}^{\pi} H\left(\omega - \frac{\Omega}{2}\right) H^*\left(\omega + \frac{\Omega}{2}\right) e^{j\omega t} d\omega. \end{aligned} \quad (7.45)$$

Here  $\chi_0(t, \Omega)$  - function of the uncertainty/indeterminacy of single sample on axis  $t=0$ .

## Chapter 8.

## SYNTHESIS OF FREQUENCY-MODULATED SIGNALS.

The methods of the synthesis of signals with the frequency modulation are worked out comparatively fully. This type of serrated signals found use in the radar of earlier than others [39], and to questions of the optimization of ChM signals are devoted many works. Let us point out, in particular, the article of Kay, etc. [36], where is for the first time published the asymptotic method of synthesis according to the assigned autocorrelation function. This method later was made more precise and was developed with a number of the authors [7, 29]. It is possible to note also the work of Cook and Pacilillo [18], of the indicated the need for refinement asymptotic decisions and proposed the method of obtaining more accurate results. Certain special forms of ChM signals were traced in works [13, 57].

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Primary task of present chapter lies in the fact that, after rethinking the known methods of synthesis of ChM signals, to show that in fact these methods are based on the hypothesis of proximity. This will make it possible to introduce the series/row of

refinements, and, which is more important, to work out further analogous methods for the signals of other types.

### 8.1. Approximations/approaches on the set ChM of signals.

Further by many permissible signals  $X$  we understand many signals with the frequency modulation, which have the final duration  $T$ ,

$$x(t) = B(t)e^{j\varphi(t)}; |t| < T/2. \quad (8.1)$$

It is assumed that amplitude envelope  $B(t)$  is fixed/recorded, for example, it has square form<sup>1</sup>

$$B(t) = \begin{cases} 1/\sqrt{T} & \text{при } |t| < T/2; \\ 0 & \text{при } |t| > T/2. \end{cases} \quad (8.2)$$

Key: ① with.

FOOTNOTE 1. Amplitude  $1/\sqrt{T}$  provides standardization on the energy.  
ENDFOOTNOTE.

The law of phase modulation  $\varphi(t)$  is arbitrary, arbitrary also the law of a change in the instantaneous frequency

$$\omega_c(t) = \frac{d\varphi}{dt}.$$

One signal of set  $X$  differs from another in terms of the structure of frequency (phase) changes. In certain cases we will consider arbitrary also duration of ChM signal  $T$  (retaining the shape of the envelope of the given one). We will respectively distinguish set of ChM signals of fixed period of time  $X_T$  from the set of ChM signals of arbitrary duration  $X$ . Is obvious,  $X_T \subset X$ . Let there be in

the space of signals  $H$  certain desired signal

$$y(t) = A(t)e^{j\phi(t)}. \quad (8.3)$$

Let us find ChM signal  $x(t)$  of duration  $T$ , which ensures best approximation to assigned  $y(t)$ , i.e., will solve the task of approximation on set  $X_T$ .

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As usual, this task is reduced to finding of the coefficient of proximity between set  $X_T$  and signal  $y$

$$C(X_T, y) = \max_{x \in X_T} \operatorname{Re} \int_{-\infty}^{\infty} x(t) y^*(t) dt. \quad (8.4)$$

Let us demonstrate the following theorem:

a) Best quadratic approximation to signal  $y(t)$  is provided on set  $X_T$  with the coincidence of the phase functions of the unknown and approximated signals

$$\varphi(t) = \Phi(t); \quad (8.5)$$

b) If signal  $y(t)$  is different from zero in any time interval of final measure with  $|t| < T/2$ , signal of best approximation on set  $X_T$  only;

c) Shortest distance  $d(X_T, y)$  and coefficient of proximity  $C(X_T, y)$  depend only on amplitude envelopes and are given by the expressions

$$d^2(X_r, y) = \int_{-\infty}^{\infty} [A(t) - B(t)]^2 dt; \quad (8.6)$$

$$C(X_r, y) = \int_{-T/2}^{T/2} A(t) B(t) dt. \quad (8.7)$$

For the proof let us substitute values (8.1) and (8.3) in (8.4)

$$\begin{aligned} C(X_r, y) &= \max_{\varphi} \operatorname{Re} \int_{-T/2}^{T/2} A(t) B(t) e^{j[\varphi(t) - \Phi(t)]} dt = \\ &= \max_{\varphi} \int_{-T/2}^{T/2} A(t) B(t) \cos[\varphi(t) - \Phi(t)] dt. \end{aligned}$$

Of theorem conditions are here assigned the positive functions  $A(t)$  and  $B(t)$ , and also phase  $\Phi(t)$  of signal  $y(t)$ . Maximization is produced according to the functions  $\varphi(t)$ , which differ one signal of set  $X_r$  from another. It is obvious, maximum reaches when  $\varphi(t) = \Phi(t)$  for all values of  $t$ , at which  $A(t) \neq 0$  and  $B(t) \neq 0$ .

FOOTNOTE 1. With exception of an arbitrary multitude of the zero measure (see note on page 188). ENDFOOTNOTE.

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Case  $B(t) = 0$  is not of interest, since in this case the phase of signal  $x(t)$  is not determined. The aforesaid proves all confirmations of theorem.

Let us focus attention on the similarity of this theorem to the

theorem of §7.2. In fact are examined completely analogous tasks, but one of them is treated in the frequency domain (is determined the optimum phase spectrum of signal with the assigned amplitude), and other - in the temporary/time (is determined the optimum law of phase modulation with assigned amplitude envelope).

If duration of ChM signal previously is not assigned, but it is determined with the synthesis so that would be obtained the best approximation on set  $X$ , additionally is produced maximization on  $T$ . Taking into account (8.7), in this case we obtain

$$C(X, y) = \max_T \int_{-T/2}^{T/2} A(t) B(t) dt. \quad (8.8)$$

8.2. Synthesis ChM of signals according to the function of uncertainty/indeterminacy.

Let us use these results for the synthesis of ChM signal according to the assigned realizable function of uncertainty/indeterminacy. In §7.1 it was shown that for this it is necessary to determine signal  $x_{opt}$ , nearest to many desired signals  $Y$ , moreover the latter is determined in the form:  $y \in Y$ , if

$$y(t) = s(t) e^{j\varphi_0},$$

where  $s(t)$  - the signal, which realizes the assigned function of uncertainty/indeterminacy  $x_s(t, \Omega)$ , and  $\varphi_0$  - arbitrary initial phase. One signal of set  $Y$  differs from another only in terms of this

initial phase.

After fixing certain signal  $y \in Y$ , possible, using the previous theorem, to determine the nearest to it signal of set  $X$ , and then, varying initial phase  $\phi_0$  (being moved on set  $Y$ ), to obtain the shortest distance  $d_{min}$ <sup>1</sup>.

FOOTNOTE <sup>1</sup>. Let us note that this order of the minimization of distance reverse used in §7.1. ENDFOOTNOTE.

But, as it is clear from (8.6), distance from signal  $y$  to set  $X$  does not depend on initial phase  $\phi_0$ .

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This means that many desired signals  $Y$  equidistantly in this task with the set of ChM signals  $X_T$ , all signals  $y \in Y$  they are placed at equidistances from  $X_T$ . Therefore initial phase  $\phi_0$  can be selected arbitrary, for example, to assure  $\phi_0 = 0$ , and this will not influence the quality of approximation/approach.

Let us consider further a specific example. As it was shown in Chapter 6, one of the signals of optimum ones from the point of view of the concentration of the body of uncertainty/indeterminacy, is

LFM signal with gaussian envelope

$$y(t) = \frac{\pi^{-1/4}}{\sqrt{2}} e^{-\pi/2 t^2} e^{-j\pi t^2}. \quad (8.9)$$

Here the first factor provides standardization on the energy. This signal is difficult to achieve, since is required a deep amplitude modulation. Therefore let us synthesize ChM signal with rectangular envelope (8.2), which gives best approximation to a function of the uncertainty/indeterminacy of signal (8.9).

In accordance with the previous theorem the unknown signal must have with rectangular envelope of  $B(t)$  the same law of phase modulation, i.e.,

$$x(t) = \frac{1}{\sqrt{T}} e^{j\pi t^2}, |t| \leq \frac{T}{2}. \quad (8.10)$$

If duration  $T$  is assigned previously, then on this synthesis is finished. But if it is necessary to determine the optimum value of  $T$ , we come to the maximization of value (8.8):

$$C(X, y) = \frac{\pi^{-1/4}}{\sqrt{2}} \int_{-T/2}^{T/2} e^{-\pi/2 t^2} dt = \left(\frac{\pi}{2}\right)^{1/4} \frac{\Phi(z)}{\sqrt{z}} = \max. \quad (8.11)$$

Here  $z = T/2\sqrt{2}$ , and  $\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$  - error function.

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Differentiating (8.11), we come to the equation

$$\Phi(z) = \frac{4}{\sqrt{\pi}} z e^{-z^2}.$$

which is satisfied for  $z \approx 1$ . Thus, the optimum value  $T$  comprises

$$T_{opt} \approx 2\sqrt{2}\tau. \quad (8.12)$$

With such a  $T$  the coefficient of proximity attains possible maximum. This maximum it is not difficult to compute according to formula (8.11), after assuming  $z=1$ :

$$C(X, Y) = \max_r C(X_r, Y) = \left(\frac{\pi}{2}\right)^{1/4} \Phi(1) \approx 0.95.$$

Fig. 8.1 shows the dependence of the coefficient of proximity of the duration of the approximating signal  $T$ ; along the axis of abscissas is deposited/postponed dimensionless quantity  $z = T/2\sqrt{2}\tau$ . Is there depicted bell-shaped envelope of the assigned signal and optimum rectangular envelope of duration  $T_{opt}$ . The instantaneous frequency of these signals is changed equally, according to the linear law.

The distance between the given one and that approximating by signals it composes

$$d_{min}^2 = 2[1 - C(X, Y)] = 2[1 - 0.95] = 0.1.$$

The standard deviation of the corresponding functions of uncertainty/indeterminacy is easy to count according to formula (7.9):

$$d_{min}^2(\gamma_s, \gamma_d) = 2[1 - C(X, Y)] = 2[1 - 0.95^2] = 0.2.$$

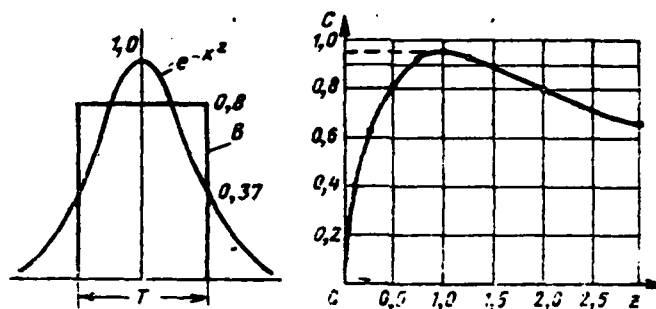


Fig. 8.1.

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As is known, the functions of the uncertainty/indeterminacy of LFM signals with gaussian and rectangular envelope, are sufficiently close. For these signals the body of uncertainty/indeterminacy has the elongated, elliptic form. The inclination/slope of the axis of ellipse depends on rate of change in the frequency and with the done approximation is identical. Some differences are in the fine structure of the body of uncertainty/indeterminacy, in particular, in the fact that the sections along the axis of frequency are represented as different functions. For rectangular envelope this section has the form

$$\chi(0, \Omega) = \frac{\sin \Omega T/2}{\Omega T/2},$$

while for gaussian envelope

$$\chi(0, \Omega) = e^{-\Omega^2/4}.$$

The optimum selection of duration  $T$  provides, in particular, the possible approximation/approach of these sections.

Taking into account as a whole the general/common/total structure of the body of uncertainty/indeterminacy, it is possible to say that the character of frequency modulation has the prevailing value. The method of synthesis examined leads to the identical frequency modulation for the assigned and approximating signals and thus provides the proximity of the functions of uncertainty/indeterminacy. Is retained only the difference in the envelopes, caused by the structure of the permitted ChM signals.

Although the obtained results are comparatively trivial, one should emphasize that we found the completely strict method of synthesis of ChM signals from the assigned realizable function of uncertainty/indeterminacy. Method provides the minimum of quadratic error for the arbitrary given envelope shape and is realized very simply. This simplicity of decision is caused, obviously, by the fact that we realize synthesis according to the realizable function of uncertainty/indeterminacy, which is found preliminarily without the limitations to many permissible signals. For finding this function of uncertainty/indeterminacy it is possible to use the methods, presented in Chapter 5.

But is feasible another, generally speaking, more correct method of synthesis. With this ChM signal  $x(t)$  it is selected so that its function of uncertainty/indeterminacy  $\chi_x(t, \Omega)$  would implement best approximation to the arbitrary desired function  $F(t, \Omega)$ .

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The latter is not realized as the function of uncertainty/indeterminacy and it is usually assigned only on the modulus/module. In particular, in work [85] is used the following criterion:

$$\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [|F(t, \Omega)|^2 - |\chi_x(t, \Omega)|^2]^2 dt d\Omega = \min.$$

The function of uncertainty/indeterminacy  $\chi_x(t, \Omega)$  complicatedly depends on phase  $\phi(t)$ , determining (with assigned envelope) ChM signal  $x(t)$ . The minimization of functional  $\xi$  according to the functions  $\phi(t)$  is the target of calculation. The analytical minimizations of so complicated a functional, it is understood there does not exist. In [85] is used iterative gradient method. Calculation is characterized by the large space of calculations. Furthermore the nonlinear functional being investigated has local extrema, and also characteristic "ravine" structure. Apparently, simpler method examined above of synthesis is advisable, at least, for obtaining the initial approximation/approach, which then can be

made more precise with the help of the iterations.

### 8.3. Synthesis ChM of signals according to the autocorrelation function.

The known methods of synthesis of ChM signals have by main target an approximation/approach to the assigned autocorrelation function. This is connected with the fact that ChM signals frequently are used for measuring only the range of the targets when the expected Doppler effects are low. The signals with linear ChM (and close to them) possess also that special feature/peculiarity, that the sections of the body of uncertainty/indeterminacy at the different values  $\Omega$  are similar to each other. This provides permission/resolution in the range even when Doppler rates are relatively great. During this use main role again plays only the form of autocorrelation function.

As shown in §7.2, synthesis according to the realizable autocorrelation function  $R(t)$  is reduced to the minimization of the distance between many permissible signals  $X$  and many desired signals  $Y$ , the latter having the assigned autocorrelation function  $R(t)$ , i.e., the assigned amplitude spectrum  $a(\omega)$ :

$$\tilde{y}(\omega) = a(\omega) e^{-j\omega t} \quad (8.13)$$

Phase spectrum  $\alpha(\omega)$  is arbitrary, this differs one signal of set  $Y$  from another. Here we examine the task of synthesis in space  $L^2$ , so that approximation/approach is understood in the sense of criterion (7.17).

Let us give two formulations of the task indicated. Let us fix first certain permissible ChM signal

$$x(t) = B(t)e^{j\varphi(t)} \quad (8.14)$$

and it is determined distance from this signal to set  $Y$ . The corresponding theorem was by us proved in §7.2. According to this theorem, to the assigned amplitude spectrum  $\alpha(\omega)$  it is necessary to ascribe the phase spectrum of signal (8.14), i.e., to place

$$\alpha(\omega) = \beta_x(\omega),$$

where  $\beta_x(\omega)$  is determined from the expression

$$\tilde{x}(\omega) = b_x(\omega) e^{-j\beta_x(\omega)} = \int_{-T/2}^{T/2} B(t) e^{j[\varphi(t) - \omega t]} dt. \quad (8.15)$$

The corresponding coefficient of proximity  $C(x, Y)$  depends only on amplitude spectrum  $b_x(\omega)$  and it is given by formula (7.15). Then in order to obtain shortest distance  $d_{min}$  it is necessary to maximize the coefficient of proximity also in signals  $x(t)$ .

As a result we come to the following variation problem. It is

necessary to determine the function  $\phi(t)$ , which gives maximum to value

$$C(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b_x(\omega) d\omega = \max. \quad (8.16)$$

where  $b_x(\omega)$  is determined in the form

$$b_x(\omega) = \left| \int_{-T/2}^{T/2} B(t) e^{j[\phi(t) - \omega t]} dt \right|. \quad (8.17)$$

The function  $\phi(t)$ , which satisfies these conditions, and is the unknown law of phase modulation.

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We will now obtain the same distance  $d_{min}$  functioning in other order. Let us fix arbitrary signal  $y(t) \in Y$  and it is determined distance from it to set  $X$ . According to theorem of §8.1 for this we must ascribe to assigned envelope of  $B(t)$  phase function  $\Phi(t)$  of signal  $y$ , i.e., to assume

$$\phi(t) = \Phi(t).$$

Here function  $\Phi(t)$  is determined in accordance with (8.13):

$$y(t) = A(t) e^{j\phi(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) e^{-j[\phi(t) - \omega t]} d\omega. \quad (8.18)$$

The coefficient of proximity  $C(X, y)$  depends only on amplitude envelope  $A(t)$  - and it is given by formula (8.7) or (8.8). In order to obtain further distance  $d_{min}$ , it is necessary to lead minimization also

on signals  $y$ . As a result we come to the maximization of value

$$C(X, y) = \int_{-T/2}^{T/2} B(t) A(t) dt = \max, \quad (8.19)$$

where

$$A(t) = \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) e^{-j[\alpha(\omega) - \omega t]} d\omega \right|. \quad (8.20)$$

Unknown is here function  $\alpha(\omega)$  - the phase spectrum of optimum generating signal  $y_{opt}(t)$  (see §1.8). If is determined phase spectrum  $a_{opt}(\omega)$ , which satisfies conditions (8.19), (8.20), then further is located most generating signal, for which is used the Fourier transform (8.18). Obtained phase function  $\Phi_{opt}(t)$  is assigned finally by assigned by amplitude envelope, that also gives the unknown ChM signal:

$$x_{opt}(t) = B(t) \exp [j\Phi_{opt}(t)].$$

In the latter/last transition it is taken into consideration, that signal  $x_{opt}$  is the element of set  $X$ , nearest to  $y_{opt}$ . Therefore we applied theorem of §1.8 to determination of  $x_{opt}$  with respect to  $y_{opt}$ . Let us remember that in §1.8 this method of synthesis was named the synthesis of the optimum generating signal with the subsequent approximation.

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From that presented it is clear that both approaches to the

synthesis lead to the similar variational problems. Conditions (8.16), (8.17) and (8.19) (8.20) are characterized by only the fact that the temporary/time and frequency dependences vary by roles<sup>1</sup>.

FOOTNOTE <sup>1</sup>. Thus it was obtained because sets X and Y were determined in this task analogously, besides one of them is assigned in the frequency domain, and another - in the temporary/time. With the synthesis of FM signals we will meet with the larger difference in the structures of the sets indicated and then these methods of synthesis will be essentially distinguished. ENDFOOTNOTE.

Unfortunately, does not succeed in proposing the direct method of deciding the variational problems indicated. Main obstruction lies in the fact that conditions (8.17) and (8.20) contain the moduli/modules of integrals. To operate with such expressions is difficult. Will be examined below the corresponding approximation method, suitable for ChM signals with high compression when integrals indicated can be computed with the help of the principle of steady state.

However, is not difficult to construct the iterative procedure, which makes it possible to reduce step by step the distance between examined/considered by sets X and Y. This procedure completely corresponds to the method of the successive design (see §1.8 and

§1.10).

Being transmitted from certain initial signal  $x_0 \in X$ , let us first determine nearest to it signal  $y_1 = P_Y(x_0) \in Y$ . Let the distance between these signals be  $d_1$ . Then let us find signal  $x_1 \in X$ , closest to  $y_1$  and located at a distance of  $d_2$  from it. Let us further determine signal  $y_2 \in Y$ , nearest  $x_1$ , then - signal  $x_2 \in X$ , nearest  $y_2$  and so forth. It is obvious, this process leads to descending sequence of the distances

$$d_1 \geq d_2 \geq d_3 \geq \dots \quad (8.21)$$

Since this sequence is bounded below ( $d \geq d_{\min}$ ), it converges to certain limit.

Does coincide this limit with a smallest distance of  $d_{\min}$ , it depends on the form of the curves  $X$  and  $Y$ . If the minimum of distance is unique, then as a result of the fact that sequence (8.21) converges to the minimum, process unavoidably leads to the shortest distance.

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But if the curves  $X$  and  $Y$  have complicated character, there are several local minimums of distance, then process leads to some minimum, but, perhaps, not smallest of base. Success of decision depends on how closely to  $x_{opt}$  is selected the signal of the initial approximation/approach  $x_0$ . Further the method of obtaining the

initial approximation/approach in question provides (at least, in some important cases) proximity to the global minimum of distance.

The process presented is reduced to the consecutive fulfillment of two basic operations: to finding ChM signals  $x_i \in X$ , closest to some signals  $y_i$ , and to finding desired signals  $y_i \in Y$ , nearest to ChM signals  $x_i$ . Both these operations we know how to make. The first of them is determined by theorem of §8.1: in order to obtain  $x_i(t)$ , nearest to  $y_i(t)$ , it is necessary for assigned amplitude envelope  $B(t)$  to ascribe phase  $\Phi_i(t)$  of signal  $y_i(t)$ . The second operation is determined by theorem of §7.2: in order to obtain signal  $y_i$ , nearest to  $x_i$ , necessary to determine phase spectrum  $\beta_x(\omega)$  of signal  $x_i$  and to ascribe this phase spectrum to the assigned amplitude spectrum  $a(\omega)$ . As a result will be formed spectrum  $\tilde{y}_i(\omega)$  of the unknown signal, and further it suffices to fulfill Fourier transform in order to switch over to the function of time.

Thus, the process in question is reduced to the successive adding of phase functions in the temporary/time and in frequency domains respectively. This adding of phase is a design to the appropriate sets  $X$  and  $Y$ .

We arrived at one of the known methods of synthesis. This method was proposed in 1959 by Tartakovskiy for the equivalent task of the

synthesis of antennas [73]. In the application/appendix to the synthesis of ChM signals the method was modified in [7]. This method, therefore, is nothing else but the method of successive design, based, in turn, on the criterion of proximity.

In §1.10 were traced questions of the convergence of the method of successive design. It was established/installed, in particular, that an in question in this task multitude of the permitted by ChM signals  $X$  is not convex. For analogous reasons convexly and desired set  $Y$ .

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Therefore the iterative method in question, although gives monotonic decrease of the distances between the sets, can, generally speaking, not converge in the sense that the obtained successive approximations  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$  ... do not approach the specific limit. As shown in §1.10, for the convergence of iterations it is necessary that obtained ChM signals  $x_i(t)$  too would not differ from nearest to them desired signals  $y_i(t) = P_Y(x_i)$  (see conditions (1.49), (1.50)). In other words, it is required so that the initial approximation/approach would provide sufficiently low distance between the sets, proximity to the optimum. This again indicates the need of obtaining good initial approximations/approaches.

FOOTNOTE 1. Below we will see, that the asymptotic decision (utilized as the initial approximation/approach) makes it possible to obtain for ChM signals with the large compression the arbitrarily low distance between  $X$  and  $Y$ . This provides the convergence of method. Convergence was confirmed also based on specific examples by the calculations of L. B. Tartakovskiy [74]. ENDFOOTNOTE.

The proved above theorems make it possible to establish also that during the completely acceptable limitations occurs the uniqueness of approximations/approaches in each stage of iterations. Let in the course of iterations be obtained certain signal  $x_i(t)$ . Because of theorem of §7.2, the transition from  $x_i$  to  $y_i$  it is realized by an only form, if spectrum  $\tilde{x}_i(\omega)$  is different from zero in each finite frequency range. But ChM signal  $x_i(t)$  has the final duration  $T$ . Consequently, spectrum  $\tilde{x}_i(\omega)$  is the whole analytic function which can take zero values only at the isolated points of the axis of frequencies (on the null set), and the condition of uniqueness is satisfied. Let us consider now transition from  $y_i(t)$  to following ChM signal  $x_{i+1}(t)$ . By the force of theorem of §8.1 this transition is unique, if signal  $y_i(t)$  is different from zero in any finite time interval with  $-T/2 < t < T/2$ . This condition is satisfied in many instances. For example, it is possible to assume that the

assigned amplitude spectrum  $a(\omega)$  is limited by the arbitrarily large, but final frequency band:

$$a(\omega) \equiv 0 \quad \text{with} \quad |\omega| > \Omega. \quad (8.22)$$

Then any signal  $y \in Y$  is a whole analytic function, and the condition of uniqueness again is satisfied.

One should however emphasize that for the practical use of a method questions of convergence and uniqueness of iterations have nevertheless secondary value.

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In §7.2 it was shown that precisely distance  $d_i = \|x_i - \tilde{y}_i\|$  is the satisfactory measure for the approximation/approach of autocorrelation functions, see formula (7.23). Therefore the monotonic decrease of distances  $d_i$  - the convergence of process on the functional, which does not require any conditions, already provides the practical applicability of iterations.

If we take into account this observation, one should recognize that the principal value has a question not about that, descent iterations or they diverge, but only about that, they do lead to

shortest distance  $d_{min}$  or the maximum distance  $d^{(0)}$  is more than  $d_{min}$ . In the latter case signal  $x^{(0)}$ , found as a result of iterations, will not be optimum from the point of view of the approximation/approach of autocorrelation functions. As it was noted, a similar situation was possible, since there are several local minimums of distance.

Consequently, the fundamental condition, which ensures the efficiency of method, is the selection of initial signal, it is sufficient close one to the optimum; are necessary the special methods, which make it possible to find rough approximation, and it subsequently it is possible to make more precise via iterations.

One of such approximation methods is examined further. This method, based on the asymptotic solution of the formulated problem of synthesis, is used for the most important virtually case of ChM signals with the large compression has also independent value.

#### 8.4. ChM signals and the method of steady state.

Further the method of synthesis in question is based on the approximation calculus of integrals of the rapidly oscillating functions - principle of steady state. Let us consider the integral

$$I = \int_a^b F(x) e^{imf(x)} dx.$$

Functions  $F(x)$  and  $f(x)$  are assumed to be those slowly varying, i.e., functions themselves and their derivatives are of the order of one. If parameter  $m$  is sufficiently great ( $m \gg 1$ ), then the function

$$e^{jm f(x)} = \cos m f(x) + j \sin m f(x)$$

is rapidly oscillating.

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The integrand can be likened to the high-frequency oscillation, modulated on the amplitude and the phase. Integral  $J$  gives the "constant component" of this oscillation and the less, the greater  $m$ . This it is possible to explain by the fact that neighboring half-waves of oscillation - positive and negative - almost compensate each other and is made a very low contribution to value  $J$ . However, the compensation for adjacent half-waves stops ineffective near the points steady state, determined by the condition

$$f'(x_0) = 0. \quad (8.23)$$

At such points "instantaneous frequency" it becomes equal to zero and oscillating process ceases (it is more precise, it stops). As a result the main contribution to the integral introduce precisely the points of steady state, if they exist in the interval  $(a, b)$ . For calculating the contribution from the stationary point it suffices to take into account behavior  $F(x)$  and  $f(x)$  in its vicinity. This leads to dependence [6]:

$$\int_a^b F(x) e^{imf(x)} dx = \sqrt{\frac{2\pi}{|mf''(x_0)|}} f'(x_0) \times \\ \times \exp \left\{ j \left[ mf'(x_0) \pm \frac{\pi}{4} \right] \right\} + O(1/m). \quad (8.24)$$

It is here assumed that in the interval (a, b) is only one stationary point<sup>1</sup>, which satisfies condition (8.23), for which  $|mf''(x_0)| \gg 1$ .

FOOTNOTE 1. If there are several stationary points, it is necessary to take the sum of corresponding components/terms/addends.

ENDFOOTNOTE.

In the index is taken plus sign with  $f''(x_0) > 0$  and minus sign is - with  $f''(x_0) < 0$ .

Correction term  $O(1/m)$  gives the estimation of error in this formula: error considers, in particular, contribution from other sections of range of integration in which there are no stationary points. Therefore evaluation/estimate  $O(1/m)$  is valid also for entire integral, if there are no points of steady state.

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Let us compute on this base the spectrum of ChM signal (8.1)

$$\tilde{x}(\omega) = \int_{-T/2}^{T/2} B(t) e^{j[\varphi(t) - \omega t]} dt.$$

Let the instantaneous frequency be changed for time T within the limits  $-\Omega \leq \omega_c(t) \leq \Omega$ , so that the deviation of frequency comprises  $2\Omega$ . Since the instantaneous frequency is derivative of phase, it is possible to register

$$\varphi(t) = \int_0^t \omega_c(t) dt = \Omega \int_0^t \gamma(t) dt = \Omega T \int_0^1 j(\eta) d\eta = \pi m \varphi_1(\eta).$$

Here  $\eta = t/T$  - dimensionless time,  $\gamma(\eta)$  and  $\varphi_1(\eta)$  - the function of the order of one and  $m = \Omega T/\pi$  - contraction coefficient.

As a result it is obtained

$$\tilde{x}(\omega) = T \int_{-1/2}^{1/2} B(\eta) e^{j\pi m(\varphi_1(\eta) - \nu \eta)} d\eta,$$

where  $\nu = \omega/\Omega$  - dimensionless frequency. After using to this integral formula (8.24) and being returned after this to initial to the variable/alternating  $t$  and  $\omega$ , we find:

$$\tilde{x}(\omega) = \sqrt{\frac{2\pi}{|\omega'_c(t_0)|}} B(t_0) \times \\ \times \exp\left\{j\left[\varphi(t_0) - \omega t_0 \pm \frac{\pi}{4}\right]\right\} + O(1/m). \quad (8.25)$$

Here moment/torque  $t_0$  is defined by equation (8.23), which, as can easily be seen, has a form

$$\varphi'(t_0) - \omega = 0$$

or, which is equivalent,

$$\omega_c(t_0) = \omega. \quad (8.26)$$

This relationship/ratio makes simple physical sense. It together with (8.25) shows that in accordance with the principle of steady state the spectrum of ChM signal at the frequency  $\omega$  is determined, in essence, by the behavior of signal at the moment (or moments/torques) of time  $t_0$ , when instantaneous frequency passes value  $\omega$ .

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This will be coordinated by the known approximation methods, utilized in the radio engineering calculations.

Formula (8.25) assumes that there is only one stationary point in the interval of integration; therefore it is applicable for the signals whose instantaneous frequency vary monotonically - it grows or decreases. With a change in the frequency  $\omega$  equation (8.26) is

satisfied at different values of  $t_0$ , stationary point is moved in the duration of signal. Thus far frequency  $\omega$  is selected in the limits of band  $\pm\Omega$ , spectral function  $x(\omega)$  is determined by the moment/torque of time  $t_0=t_0(\omega)$ . In other words, in this frequency domain the spectrum is approximately determined not by signal as a whole, but it is local, by the moment/torque of time  $t_0$ , which depends on  $\omega$ . This connection/communication between the instantaneous values of frequency and time is the base of further calculation.

If spectral frequency  $\omega$  is selected out of the band  $\pm\Omega$ , equation (8.26) is satisfied not with what  $t$  in the interval  $(-T/2, T/2)$ , i.e., stationary points are absent. Formula (8.25) in this case becomes meaningless. However, spectral function in this region is low, it is estimated at value of 0 (1/m) and can in the first approximation, be disregarded for the signals with the large compression. The fundamental portion of energy of such signals is concentrated in the band  $\pm\Omega$ , i.e., frequency domain, by the running instantaneous frequency.

Let us consider in more detail the structure of spectral function in the fundamental region  $(-\Omega, \Omega)$ . For the amplitude spectrum we have from (8.25)

$$b_x(\omega) = |\tilde{x}(\omega)| = \sqrt{\frac{2\pi}{|\omega'_x(t_0)|}} B(t_0) + O(1/m). \quad (8.27)$$

4

This relationship/ratio shows that the amplitude spectrum depends on the signal amplitude at moment/torque  $t_0$  and on rate of change in the frequency for this moment/torque. The greater the rate of modulation  $\omega'_c(t_0)$ , the less the level of spectral function. This dependence is confirmed by known physical considerations [6].

For the phase spectrum we have respectively

$$\beta'_x(\omega) = -[\varphi(t_0) - \omega t_0 \pm \pi/4]. \quad (8.28)$$

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Let us show that derivative  $\beta'_x(\omega)$  is a monotonic function of frequency. Actually/really, taking into account (8.26), it is not difficult to obtain

$$\begin{aligned} \frac{d\beta'_x}{d\omega} &= \frac{d\beta'_x}{dt_0} \cdot \frac{dt_0}{d\omega} = - \frac{d}{dt_0} \left[ \varphi(t_0) - \omega_c(t_0)t_0 \pm \frac{\pi}{4} \right] \frac{1}{\omega'_c(t_0)} = \\ &= - [\omega_c(t_0) - \omega_c(t_0) - t_0 \omega'_c(t_0)] \frac{1}{\omega'_c(t_0)} = t_0. \end{aligned}$$

Consequently, again applying (8.26), we find

$$\beta''_x(\omega) = \frac{dt_0}{d\omega} = \frac{1}{\omega'_c(t_0)}.$$

For the signal with the monotonically changing frequency value  $\omega'_c(t_0)$  does not change sign on the entire duration T. Therefore  $\beta''_x(\omega)$  also does not reverse the sign, that also proves the expressed confirmation.

Let us focus attention on the following property of symmetry. According to the condition, the law of a change in the phase  $\phi(t)$  is such, that its derivative  $\phi'(t) = \omega_c(t)$  is monotone. Phase spectrum  $\beta_x(\omega)$ , proves to be, possesses the similar property: derivative  $\beta'(\omega)$  is monotonic, besides has the same character of the change (it grows at the increasing instantaneous frequency or decreases - with that decreasing). So stand matters, at least, in the approximation/approach of the method of steady state, i.e., with the high contraction coefficients  $m^{-1}$ .

FOOTNOTE 1. On the base of the method of steady state it is possible to compute the spectrum of ChM-oscillation/vibration also in the more complicated cases, in particular, when there are several stationary points [5, 6]. But the obtained expressions are not used with the synthesis of signals due to the unwieldiness. ENDFOOTNOTE.

Let us now consider inverse problem - recovery of signal on its spectrum. We have

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) e^{-j[\omega t - \beta(\omega)]} d\omega. \quad (8.29)$$

It is assumed that the amplitude spectrum  $a(\omega)$  is limited by the final band of frequencies -  $\omega \leq \omega_0$  (however this assumption is not

essential). The derivative of the phase spectrum

$$\frac{d\alpha(\omega)}{d\omega} = \tau(\omega) \quad (8.30)$$

has a dimensionality of time.

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This value possesses the physical content. If, for example, signal  $y(t)$  is formed at the output of the corresponding forming filter, then  $\alpha(\omega)$  is the phase response of filter  $\tau(\omega)$  - group delay time for the frequency  $\omega$ . Analogous treatment is valid for the signals, which extend in the delay lines, in the dispersive media, etc. Let in the frequency interval  $(-\Omega, \Omega)$  in question the group delay time vary within the limits  $-T/2 < \tau < T/2$  <sup>1</sup>.

FOOTNOTE <sup>1</sup>. The selection of the zero time reference is arbitrary. Therefore, without entering into contradiction with the physical sense, it is permitted the negative values of delay time.  
ENDFOOTNOTE.

Then it is possible to register

$$\begin{aligned} \alpha(\omega) &= \int_0^T \tau(\omega) d\omega = T \int_0^T \tau_1(\omega) d\omega = \Omega T \int_0^1 \tau_1(v) dv = \\ &= \pi m_1(v). \end{aligned}$$

where  $v = \omega \Omega$  - dimensionless frequency  $\tau_1(v)$  and  $m_1(v)$  - the

dimensionless functions of the order of unity,  $m = \Omega T / \pi$ . Value  $m$  is great, if in the frequency interval of the variation in question of the phase  $\alpha(\omega)$  occurs to a large number of periods. After doing this assumption, we come to the integral of type (8.24) and find

$$y(t) = \frac{a(\omega_0)}{\sqrt{2\pi|\tau'(\omega_0)|}} \exp \left\{ -j \left[ \alpha(\omega_0) - \omega_0 t \pm \frac{\pi}{4} \right] \right\} + O\left(\frac{1}{m}\right). \quad (8.31)$$

It is here assumed that the delay time  $\tau(\omega)$  varies monotonically with the frequency, so that the stationary point is unique. This point is determined by the equation

$$\alpha'(\omega_0) = \tau(\omega_0) = t, \quad (8.32)$$

i.e. it depends on the current time  $t$ . Thus, the value of signal at moment/torque  $t$  is determined, in essence, by the structure of the spectrum at that frequency  $\omega_0$ , for which group delay time coincides with  $t$ .

For the signal amplitude we obtained, obviously,

$$A(t) = |y(t)| = \frac{a(\omega_0)}{\sqrt{2\pi|\tau'(\omega_0)|}} + O(1/m). \quad (8.33)$$

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Discussing analogous with previous, it is possible to show that when making these assumptions the instantaneous signal frequency coincides for the current moment/torque  $t$  with the frequency  $\omega_0$ :

$$\begin{aligned}\omega_c(t) &= \frac{d\varphi}{dt} = \frac{d}{dt} [\alpha(\omega_0) - \omega_0 t \pm \pi/4] = \\ &= \frac{d}{d\omega_0} \left[ \alpha(\omega_0) - \omega_0 \alpha'(\omega_0) \pm \frac{\pi}{4} \right] \frac{d\omega_0}{dt} = \omega_0\end{aligned}$$

and it vary monotonically in the interval  $(-T/2, T/2)$ . Consequently, in the approximation/approach of the method of steady state ChM signals with a monotonic change in the frequency possess the phase spectrum whose derivative is monotone, and, vice versa, this spectrum can be realized only by monotonic ChM signal.

Let us note one additional fact. As can be seen from (8.33), signal amplitude is low at values of  $t$  for which equation (8.32) does not have a solution. This means that fundamental energy is concentrated in the interval  $(-T/2, T/2)$ , determined by the range of changes in the group delay  $\tau(\omega)$ . Consequently, value  $T$ , which depends on the structure of phase spectrum, is close to the pulse duration, and parameter  $m = \Omega T/\pi$  is a contraction coefficient. We see that the calculation of spectrum of ChM signal and the restoration/reduction of signal according to the spectrum can be carried out on the basis of the principle of steady state only for the high contraction coefficients.

#### 8.5. Asymptotic synthesis of ChM signals.

In §8.3 were proposed two methods of synthesis of ChM signals, escape/ensuing from the hypothesis proximity. The first of them (formula (8.16) - (8.17)) is reduced to the maximization the coefficient of closeness

$$C(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) b_x(\omega) d\omega = \max,$$

where  $b_x(\omega)$  - amplitude spectrum of the unknown signal  $x(t)$ . If we are bounded to signals with a monotone change in the frequency and large compression, it is possible to use approximation formula (8.27) for spectrum  $b_x(\omega)$ .

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As a result it is obtained

$$C(x, Y) = \frac{11}{V^{2\pi}} \int_{-\Omega}^{\Omega} a(\omega) \frac{B(t_0)}{V|\omega_c(t_0)|} d\omega + O(1/m). \quad (8.34)$$

Are here undertaken final integration limits, since the dominant term of formula (8.27) is suitable only in the band  $(-\Omega, \Omega)$ . Out of this spectrum band available estimate  $O(1/m)$ , that also gives the appropriate correction in (8.34). Let us recall that the moment/torque of time  $t_0$  is monotonic function  $\omega$ . This dependence is determined by equation (8.26). Since  $t_0$  and  $\omega$  are connected, it is possible to pass in integral (8.34) to the variable/alternating  $t_0$ . Accepting for concreteness  $\omega_c' > 0$ , i.e. instantaneous frequency it

grows, from (8.34) and (8.26) it is not difficult to obtain:

$$C(x, Y) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} B(t) a\{\omega_c(t)\} \left[ \frac{d\omega_c(t)}{dt} \right]^{1/2} dt + O(1/m), \quad (8.35)$$

moreover in the latter/last expression is omitted index in the variable/alternating of integration  $t$ . For the research of the obtained expression for the maximum we will use Schwarz-Buniakowski. Disregarding the correction term, we have

$$\begin{aligned} C^2(x, Y) &\leq \int_{-T/2}^{T/2} B^2(t) dt \cdot \frac{1}{2\pi} \int_{-T/2}^{T/2} a^2(\omega_c) \frac{d\omega_c}{dt} dt = \\ &= \int_{-T/2}^{T/2} B^2(t) dt \cdot \frac{1}{2\pi} \int_{-\Omega}^{\Omega} a^2(\omega) d\omega. \end{aligned}$$

The first integral is energy of signal  $x(t)$  and it is equal to unity by standardization strength. The second integral exists, strictly speaking, the part of the energy of signal  $y(t)$ , included in the band  $(-\Omega, \Omega)$ , but it also it is very close to unity, since residual energy decreases during the expansion of band, at least as  $O(1/m^2)$ .

Consequently, without introducing the further error (by order of value), it is possible to consider that the right side of the latter/last inequality gives face side of the coefficient of proximity  $C(x, Y)$ .

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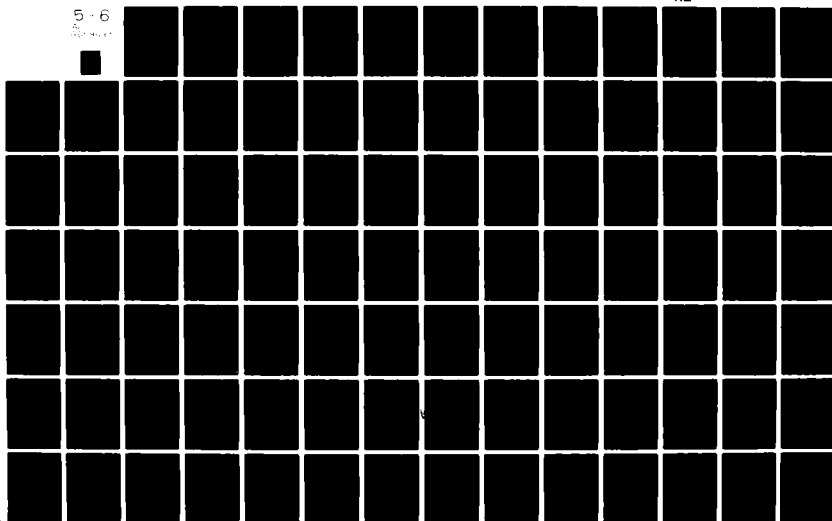
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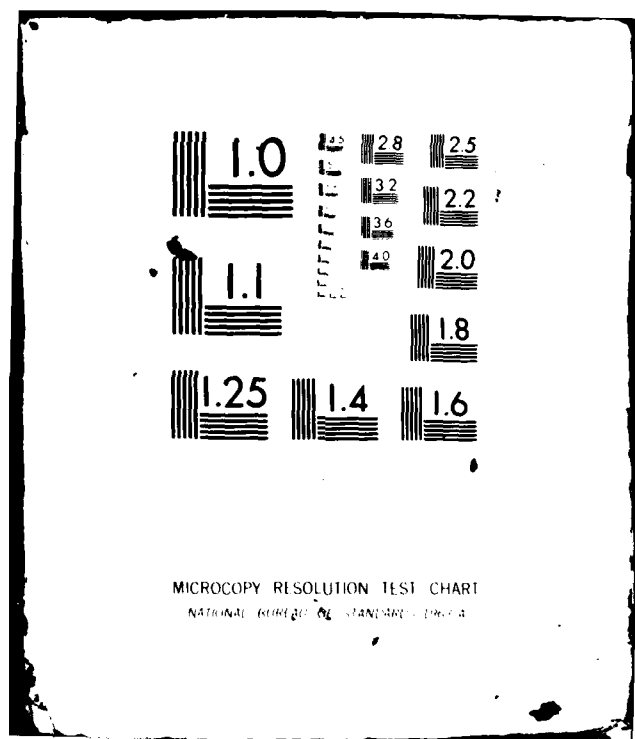
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This face side is reached, if in the relationship/ratio of Schwarz - Buniakowski occurs equal sign, i.e., when the factors of integrand in (8.35) are proportional

$$B(t) = \gamma \frac{a(\omega_0)}{\sqrt{2\pi}} \left[ \frac{d\omega_0}{dt} \right]^{1/2}.$$

So it is easy to check by straight/direct substitution, factor of proportionality  $\gamma$  in the optimum case is equal to one. As a result we come to the differential equation<sup>1</sup>

$$B^2(t) dt = \frac{1}{2\pi} a^2(\omega_0) d\omega_0. \quad (8.36)$$

being determining the optimum law of the frequency modulation of the unknown ChM signal.

FOOTNOTE 1. If instantaneous frequency  $\omega_0(t)$  decreases, and it does not grow, the right side of the equation reverses the sign. This does not lead to the essential differences. ENDFOOTNOTE.

Equation (8.36) is the base of calculation in the series/row of works according to the synthesis of ChM signals [7, 29, 36, 39]. Our conclusion/output shows that this most important method of synthesis is based, in fact, to the criterion of proximity and directly it follows from the appropriate task of the minimization of distance.

Let us now consider the second method, indicated in §8.2.

According to (8.19), (8.20) we must maximize the coefficient of the proximity

$$C(X, y) = \int_{-T/2}^{T/2} A(t) B(t) dt,$$

selecting  $A(t)$  - amplitude envelope of the generating signal. For approximation calculus  $A(t)$  we will use formula (8.33).

This it gives

$$C(X, y) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} B(t) \frac{a(\omega_0)}{\sqrt{\tau'(\omega_0)}} dt + O(1/m).$$

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Here frequency  $\omega_0$  is connected with the current time  $t$  with equation (8.32), which makes it possible to switch over in the integral to variable/alternating  $\omega_0$ . Then it is obtained

$$C(X, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(\omega_0) B\{\tau(\omega_0)\} \left[ \frac{d\tau}{d\omega_0} \right]^{1/2} d\omega_0 + O(1/m).$$

This integral can be traced to the maximum, after using, as earlier, Schwarz - Bunyakowski. As a result it is clarified, that the coefficient of proximity  $C(X, y)$  is maximum with satisfaction of the condition

$$\frac{1}{\sqrt{2\pi}} a(\omega_0) = B\{\tau(\omega_0)\} \left[ \frac{d\tau}{d\omega_0} \right]^{1/2},$$

i.e.

$$B^2(\tau) d\tau = \frac{1}{2\pi} a^2(\omega_0) d\omega_0. \quad (8.37)$$

We arrived at the equation, which coincides with (8.36). Strictly, otherwise and be it could not, since both methods of the minimization of distance, examined in §8.3, must lead to one and the same optimum signal  $x_{opt}(t)$ . This conclusion/output is of interest for future reference. For ChM signals both those examined of the method of synthesis are equivalent. As noted, they are characterized by only the fact that the time and frequency vary by roles. In the asymptotic approximation/approach we could obtain the solutions by each of the methods and clearly demonstrate their identity.

It is possible to estimate the error in the asymptotic solution examined. If instantaneous frequency satisfies equation (8.36), i.e.,

$$\frac{d\omega_0}{dt} = 2\pi \frac{B^2(t)}{a^2(\omega_0)},$$

then formula (8.35) gives

$$\begin{aligned} C(X, Y) &= \max_{x \in X} C(x, Y) = \int_{-T/2}^{+T/2} B^2(t) dt + O(1/m) = \\ &= 1 + O(1/m). \end{aligned} \quad (8.38)$$

Therefore for the distance between sets X and Y is obtained the evaluation/estimate

$$d_{min}^2 = 2[1 - C(X, Y)] = O(1/m).$$

We saw in §7.2 that a root-mean-square error in the autocorrelation functions  $\delta$  depends on distance  $d_{min}$ . This dependence is given by formula (7.25):

$$\delta_{min} = \sqrt{\frac{2}{m}} d_{min}$$

therefore

$$\delta_{min} = \sqrt{\frac{2}{m}} O(1/\sqrt{m}) = O(1/m).$$

i.e. an error in the approximation/approach of autocorrelation function decreases as  $1/m$ .

Of course these evaluations/estimates are insufficient in order to determine numerical magnitude of error in this or another specific case. From further examples it is clear that the quality of approximation/approach depends on the form of the assigned spectrum  $a(\omega)$  and the envelope  $B(t)$ . This connection/communication is easily explained - indeed the task of synthesis lies in the fact that with assigned envelope to ensure the necessary amplitude of assigned envelope to ensure the necessary amplitude spectrum. Is accurate this possible not always. There is a series/row of the conditions with nonfulfillment of which it cannot be combined assigned  $B(t)$  and  $a(\omega)$ . For example, it is not possible to fulfill the rectangular spectrum with rectangular envelope, since one of these functions must be analytical.

FOOTNOTE 1. The known conditions of the "compatibility"  $a(\omega)$  and  $B(t)$  are indicated by Fowle [29]. ENDFOOTNOTE.

In our treatment of synthesis the discussion does not deal with a precise fulfillment of the assigned spectrum. We realize a best approximation to it, but the degree of this approximation/approach depends it goes without saying on structure assigned  $B(t)$  and  $a(\omega)$ .

However, obtained estimates show that with synthesis of ChM signals the error can be arbitrarily decreased, increasing the compression coefficient. This corresponds to the fact that, increasing duration of ChM signal, it is possible to fulfill the given spectrum with the low to arbitrarily fulfill the assigned spectrum with the arbitrarily low final error (has in mind the error on the average).

Of this consists one of the special features/peculiarities of the set of ChM signals in question. We will see further, that with the synthesis of FM signals occurs different picture. There there is a final limit of distance  $d_{min}$ , so that, even increasing compression, is not possible to arbitrarily improve the degree of approximation of signals.

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Correspondingly, an error in the correlation functions decreases for FM signals not as  $1/m$ , but as  $1/\sqrt{m}$ . However, a similar position is characteristic also for ChM signals under some further conditions (see §8.7).

#### 8.6. Examples.

We will use several simple examples in order to illustrate the efficiency of asymptotic synthesis. These examples are borrowed from works [7, 29, 36, 39].

Example 1. Let us consider first the case when they are assigned rectangular enveloping

$$B(t) = \begin{cases} 1/\sqrt{T} & \text{if } -\frac{T}{2} \leq t \leq +\frac{T}{2}, \\ 0 & \text{if } |t| > \frac{T}{2} \end{cases}$$

Key: (1). with.

and the bell-shaped amplitude spectrum of the form

$$a(\omega) = \frac{\sqrt{2Q}}{\sqrt{1 + \omega^2 Q^2}}, \quad -\infty < \omega < +\infty.$$

Substituting these values in (8.36), we obtain

$$\frac{dt}{T} = \pm \frac{d\omega_c Q}{\omega_c^2 (1 + \omega_c^2 Q^2)},$$

where the signs  $\pm$  correspond to rising or falling instantaneous frequency. Solution of this equation gives the required law of the frequency modulation

$$\omega_c(t) = \pm \Omega \operatorname{tg} \pi \frac{t}{T}. \quad (8.39)$$

Assuming/setting the frequency of that falling, for the instantaneous phase we obtain respectively

$$\varphi(t) = \int \omega_c dt = m \ln \cos \pi \frac{t}{T} + \varphi_0,$$

where  $\varphi_0$  - arbitrary initial phase.

Let us further determine phase spectrum  $\beta_x(\omega)$ . For this we will use formula (8.28) and dependence of  $t$  on  $\omega_c$  (or, which is the same thing,  $t_0$  from  $\omega$ ), expressed by relationship/ratio (8.32). As a result, after simple conversions it is obtained

$$\beta_x(\omega) = m \left[ \frac{1}{2} \ln \left( 1 + \frac{\omega^2}{\Omega^2} \right) - \frac{\omega}{\Omega} \operatorname{arctg} \frac{\omega}{\Omega} \right] + \beta_0,$$

where  $\beta_0$  is also arbitrary.

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In accordance with the theorem of §7.2  $\beta_x(\omega)$  there is the phase spectrum not only of signal  $x_{opt}(t)$ , but also signal  $y_{opt}(t)$  - the nearest signal of set  $Y$ , i.e.,

$$y_{opt}(\omega) = \frac{\sqrt{2/Q}}{\sqrt{1 + \omega^2/Q^2}} \times \\ \times \exp \left\{ j m \left[ \frac{1}{2} \ln \left( 1 + \frac{\omega^2}{Q^2} \right) - \frac{\omega}{Q^2} \operatorname{arctg} \frac{\omega}{Q} \right] \right\}.$$

An error in asymptotic solution can be considered now, being congruent/equating found CHM signal  $x_{opt}(t)$  with generating signal  $y_{opt}(t)$ . The latter is determined by numerical method, by Fourier transform from  $\tilde{y}_{opt}(\omega)$ . The necessary calculations are carried out in [29]. Fig. 8.2 shows the values of instantaneous frequency and signal amplitude envelope  $x_{opt}(t)$  and  $y_{opt}(t)$  (latter are noted by points). Calculation is carried out for comparatively low contraction coefficients  $m=5/\pi$  and  $m=50/\pi$ . As can be seen from figure, even for such values of  $m$  asymptotic solution gave very good approximation/approach.

Example 2. Now let us assume that both functions - enveloping and the amplitude spectrum - are assigned rectangular:

$$B(t) = \begin{cases} 1/\sqrt{T} & \text{при } -T/2 < t < +T/2; \\ 0 & \text{при } |t| > T/2; \end{cases} \\ a(\omega) = \begin{cases} 1/\sqrt{\pi/Q} & \text{при } -Q < \omega < +Q; \\ 0 & \text{при } |\omega| > Q. \end{cases}$$

Key: (1). with.

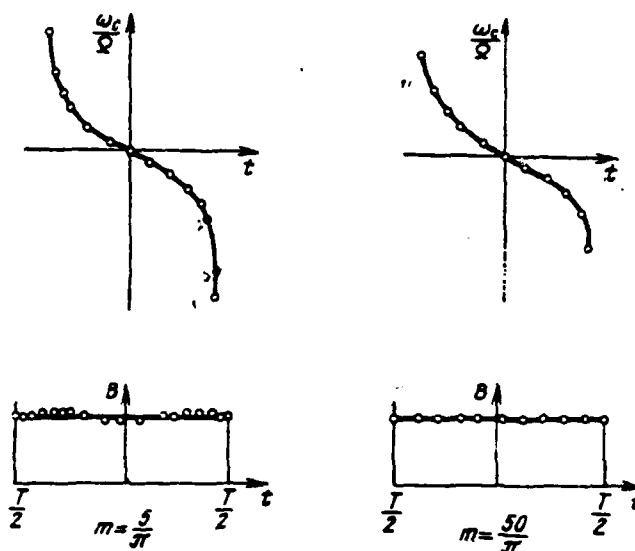


Fig. 8.2.

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It is obvious, equation (8.36) has in this case a form

$$\frac{dt}{T} = \frac{d\omega_c}{2\Omega}$$

and it leads to the linear law of modulation

$$\omega_c(t) = \frac{2\Omega}{T} t.$$

We obtained, therefore, LFM signal with rectangular envelope.

His spectrum

$$\tilde{x}(\omega) = \frac{1}{T} \int_{-T/2}^{T/2} \exp \left[ j \left( \frac{\Omega}{T} t^2 - \omega t \right) \right] dt$$

it is not difficult to express through Fresnel's integral. In particular, we have

$$b_m(\omega) = |\tilde{x}(\omega)| = \sqrt{\frac{\pi}{2Q}} \left\{ \left( C \left[ \frac{\sqrt{\pi m}}{2} \left( 1 + \frac{\omega}{Q} \right) \right] + \right. \right. \\ \left. \left. + C \left[ \frac{\sqrt{\pi m}}{2} \left( 1 - \frac{\omega}{Q} \right) \right] \right)^2 + \left( S \left[ \frac{\sqrt{\pi m}}{2} \left( 1 + \frac{\omega}{Q} \right) \right] + \right. \right. \\ \left. \left. + S \left[ \frac{\sqrt{\pi m}}{2} \left( 1 - \frac{\omega}{Q} \right) \right] \right)^2 \right\}^{1/2}.$$

Here

$$C(z) + jS(z) = \sqrt{\frac{2}{\pi}} \int_0^z e^{jz^2} dz.$$

It is possible to consider the quality of approximation/approach, being congruent/equating the assigned spectrum with that obtained. The corresponding graphs, carried out for comparatively high contraction coefficients, are depicted in Fig. 8.3 [39].

As can be seen from figures, a difference in the spectra is exhibited, mainly, near the band edges, with  $\omega = \pm Q$ . To the flat/plane part of the spectrum superimposed oscillations whose amplitude weakly is reduced with an increase in the compression. However, the region, occupied by these oscillations/vibrations, with increase of  $m$  is reduced, oscillations/vibrations "are wrung out" to the assigned boundaries  $\pm Q$ . These oscillations/vibrations call Fresnel pulsations, since they are connected with the structure of Fresnel's integrals. We see also that outside the boundary of the assigned band is a comparatively slow decay in the spectral function on which also are present Fresnel pulsations. The greater the compression, the steeper this decay. As a result, with an increase in the compression

the spectrum approaches the given one, but, in the first place, this occurs slowly, and they are necessary very more than value of  $n$  so that the distortions would be insignificant, and, in the second place, improvement occurs not due to the decrease of maximum divergences, but as a result of the contraction of the sections, in which these divergences are essential.

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One way or another, but was obtained considerably worse approximation/approach, than in the previous example, and it is necessary to be dismantled/selected at the reasons for this difference.

Using a method of steady state for calculating the spectrum, we assumed the envelope  $B(t)$  of that of slowly varying. In the case of rectangular  $B(t)$  this, of course, it is erroneous near the pulse edges where the envelope endures abrupt changes. Therefore, when the stationary point  $t_0$  is moved to the pulse edge, the utilized formulas, are incorrect. But the point of steady state coincides with one of the fronts on the band edges of frequencies, with  $\omega = \pm \Omega$ . Logically, in this region were obtained the greatest errors. Therefore it is possible to conclude that an error in the approximation/approach is substantially connected with irregularity

$B(t)$  near the fronts<sup>1</sup>.

FOOTNOTE <sup>1</sup>. If envelope has steady character, errors are obtained still smaller than in example 1. This case is examined in [29].

ENDFOOTNOTE.

However, in example 1 also was examined rectangular envelope, and errors proved to be very low even during the small compression. This forces in greater detail it will dwell on the concept of the slowly varying function.

Let there be certain envelope  $B(t)$ , which is thus far assumed to be continuous. After selecting arbitrarily moment/torque  $t_0$ , it is possible to register variation  $\Delta B$  in the form

$$\Delta B = B(t_0 + \Delta t) - B(t_0) \approx B'(t_0) \Delta t.$$

Other conditions being equal, the variation  $\Delta B$  is the greater, the greater the interval  $\Delta t$ .

We saw that during the use of a method of steady state integral value depends, mainly, on certain low vicinity of stationary point.

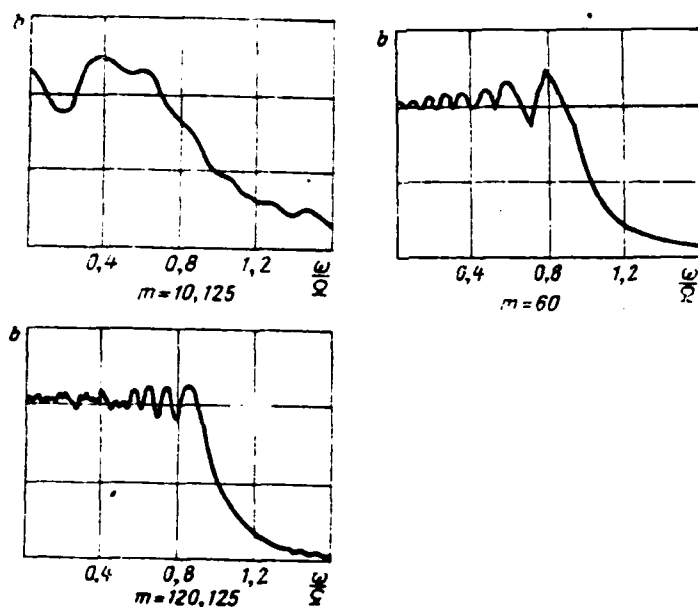


Fig. 8.3.

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The extent of this vicinity is characterized by the so-called radius of effect and depends on the form of phase function [28]. In particular, for the integral

$$\Delta t \approx \frac{1}{\sqrt{\omega_e(t_0)}}.$$

in question.

Consequently,

$$\Delta B \approx \frac{B'(t_0)}{\sqrt{\omega_e(t_0)}}.$$

Thus, inconstancy of envelope is manifested the more weakly, the greater the rate of modulation at the appropriate moment of time. Taking into account (8.27), we can register also

$$\Delta B \approx \frac{1}{\sqrt{2\pi}} \frac{B'(t_0)}{B(t_0)} b_m(\omega) \approx \frac{1}{\sqrt{2\pi}} \frac{B'(t_0)}{B(t_0)} a(\omega).$$

Here  $\omega$  - current frequency, which coincides with the instantaneous signal frequency at moment/torque  $t_0$ .

Now it is possible to clarify the difference between examples 1 and 2. In both cases the envelope is identical, so that factor  $B'(t)/B(t)$  is retained constant/invariable<sup>1</sup>.

FOOTNOTE 1. Let us recall that for simplification in the considerations is assumed the final duration of fronts, so that the derivative  $B'(t)$  is limited. ENDFOOTNOTE.

But the laws of frequency modulation essentially are distinguished. In the first example the rate of modulation grows with the approximation/approach to pulse edges, reaching infinite value with  $t = \pm T/2$ . By this is provided the bell-shaped form of the spectrum with the attenuation at the high frequencies. The previous relationships/ratios show that simultaneously is weakened/attenuated the effect of the irregularity of envelope on the edges. In example 2 rate of modulation and level of the spectrum are constant; therefore the "edge effect", connected with the pulse edges, it is exhibited to the greatest degree<sup>2</sup>.

FOOTNOTE 2. In a recent work Millet experimentally confirmed that the distortions, caused by Fresnel pulsations, they are manifested more strongly for the rectangular spectrum than for that rounded off [49]. The theoretical explanation to this was not given. ENDFOOTNOTE.

From the aforesaid it is clear that the cases examined are in a certain sense, maximum for rectangular envelope. Signal with the linear ChM (example 2), which realizes approximation/approach to the rectangular spectrum, shows the maximum divergences, connected with rectangular envelope - Fresnel pulsations.

When the assigned spectrum is rounded off, occurs a relative increase in the rate of modulation on the edges, and means, an improvement in the quality of approximation/approach. Fresnel pulsations are weakened/attenuated to the greatest degree for the signals of the type of example 1. Here the rate of modulation on the edges is infinitely great and, furthermore, the region of the greatest distortions is extruded/excluded for the infinitely high frequencies. These facts provide a good approximation/approach.

Example 3. In the previous examples the form of autocorrelation function (amplitude spectrum) was chosen arbitrarily.

However, in Chapter 2 and 4 we came to light/detected/exposed the optimum structure of the autocorrelation function, maximally concentrated in the assigned duration with the fixed/recorded width of the spectrum. In the case of the minimax criterion, optimum is Dolph-Chebyshev type function, and the corresponding form of the amplitude spectrum takes the form, see (2.26):

$$a^2(\omega) = k \frac{I_1(c \sqrt{1 - \omega^2 \Omega^2})}{\sqrt{1 - \omega^2 \Omega^2}}, \quad -\Omega < \omega < +\Omega.$$

Here  $c = \Omega T / 2 = \pi m / 2$  - value, proportional to contraction coefficient. The scale factor  $k$  must be selected so that would be satisfied the condition of normalization on energies. This it gives [7]

$$k = 2\pi \frac{c \Omega}{\operatorname{ch} c - 1}.$$

Let us recall that the level of the remainders/residues of optimum autocorrelation function is determined by the relationship/ratio

$$M = \operatorname{ch} c = \operatorname{ch} \pi m / 2,$$

which corresponds to the exponential decrease of remainders/residues in an increase in the contraction coefficient.

If, as in the previous examples, amplitude signal amplitude envelope assigned to rectangular, equation (8.36) takes the form

$$\frac{dt}{T} = \frac{c}{\operatorname{ch} c - 1} \frac{I_1(c \sqrt{1 - \omega_c^2 \Omega^2})}{\sqrt{1 - \omega_c^2 \Omega^2}} \frac{d\omega_c}{\Omega}.$$

Therefore the dependence of instantaneous frequency on the time is

given by the relationship/ratio

$$\frac{t}{T} = \frac{c}{\operatorname{ch} c - 1} \int_0^{e^{j2}} \frac{I_1(c \sqrt{1-x^2})}{\sqrt{1-x^2}} dx. \quad (8.40)$$

This dependence is shown in Fig. 8.4. The parameter is value  $M$  - the level of the remainders/residues of the assigned autocorrelation function. A decrease in the remainders/residues is provided due to an increase in the rate of modulation in pulse edges. As it follows from previous, this raises also the accuracy of the asymptotic solution.

A good approximation/approach to Dolph-Chebyshev type autocorrelation function gives, as is known, the function of Hemming, for which the form of the spectrum takes form [75]:

$$a^2(\omega) = \frac{\pi}{D} \left( 1 + g \cos \pi \frac{\omega}{D} \right); \quad -D < \omega < +D.$$

This spectrum monotonically drops to the edges of band and has jumps on the edges. The value of jump depends on parameter  $g$ . With  $g=0$  the form of the spectrum rectangular, the jumps are maximum, with  $g=1$  is obtained the cosine-squared form without the jumps, in the intermediate cases  $0 < g < 1$  the value of jump comprises  $\pi(1-g)$ .

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Changing  $g$ , it is possible to obtain approximation/approach to optimum curves for various levels of remainders/residues. The parameters of the corresponding autocorrelation functions are given

in [7, 75].

The approximation of the optimum form of the spectrum by the function of Hemming makes it possible to simplify calculation. Actually, equation (8.36) take in this case the form

$$\frac{dt}{T} = \frac{1}{2\pi} \left( 1 + g \cos \pi \frac{\omega}{\Omega} \right) d\omega.$$

As a result it is obtained

$$\frac{t}{T} = \frac{1}{2} \left( \frac{\omega}{\Omega} + \frac{g}{\pi} \sin \pi \frac{\omega}{\Omega} \right). \quad (8.41)$$

Fig. 8.5 compares the optimum law of modulation (8.40) with approximation/approach (8.41). Calculation is carried out for level of lobes/lugs  $M=40$  dB, parameter  $g$  in this case is equal to 0.85. As is evident, the curves are rather close. This confirms the expediency of using the Hemming ~~approximation~~ *approximation*.

It would be a mistake to assume that given the corresponding function of the Dolph-Chebyshev type (or an approximation to it), we will actually obtain so low a level of remainders. As it was shown, used asymptotic solution has an error of the level of the remainders/residues of the autocorrelation function of order  $O(1/m)$ . This error can prove to be considerably the assigned level, so the latter more than decreases exponentially with increase of  $m$ .

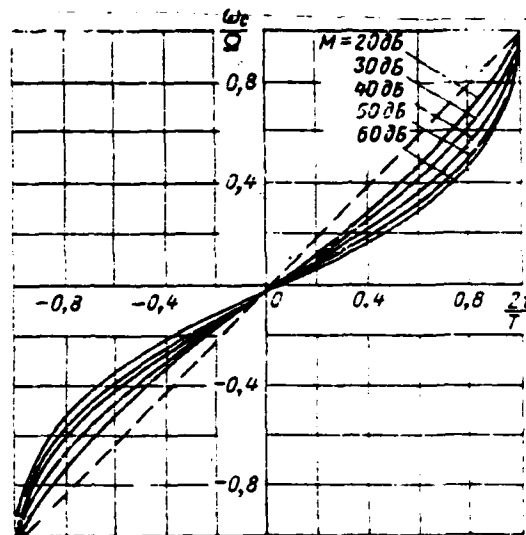


Fig. 8.4.

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In this connection is of interest the refinement of the asymptotic solution. One of the methods of refinement gives the iterative procedure, examined in §8.3. By using the asymptotic solution as zero approximation, it is possible to obtain more precise results via consecutive iterations.

Another method is based on more precise asymptotic evaluations/estimates of integral (8.24). As we saw, the neglect of "edge effect", allowed in formula (8.24), leads to the fact that the form of the spectrum only approximately corresponds to the given one.

The spectrum holds the Fresnel pulsations, not taken into consideration in the calculation, or some other inaccuracies. It is possible to compensate (at least, partially) Fresnel pulsations, outside special oscillatory addition into the law of ChM. Such "predistortions" of the structure of signal make it possible to decrease the remainders/residues. One of the methods of calculation of predistortions (by the way very approximated) is proposed by Cook and Paolillo [18].

The introduction of corrections into the law of ChM (designed on any of the methods indicated) is conjugated/combined with the known technical difficulties. Is required to satisfy complicated the law of modulation, which contains fluctuating component of changing of amplitude. Here we will not in detail trace the necessary structure of corrections.

#### 8.7. Signals with the symmetrical frequency modulation.

In the previous examination the law of a change of the instantaneous frequency it was assured to be monotonic. This limitation made it possible to use the simplest version of the method of the steady state when there is only one stationary point.

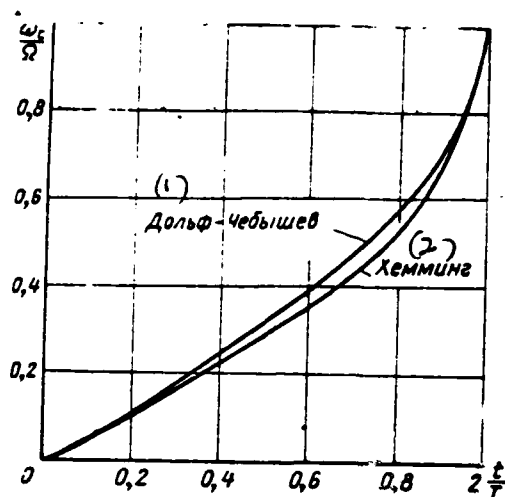


Fig. 8.5.

Key: (1). Dolph-Chebyshev. (2). Hemming.

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Limited application find also more complicated ChM signals with the nonmonotonic law of modulation. The use of similar signals expediently during the simultaneous permission/resolution in the range and speed since their functions of uncertainty/indeterminacy have the not elongated, elliptic form, but they possess more or less expressed central peak [13]. In this respect the signals in question are close to phase-keyed.

As shown below, analogy with FM signals extends somewhat

further. We will see, that the methods of synthesis of FM signals and the examined signals with the frequency modulation have much in common.

We will be bounded to the cases when instantaneous frequency is changed symmetrically relative to the middle of impulse/momentum/pulse and, furthermore, it is monotone in each half of signal. A similar signal is used, for example in the system, described in work [56].

If the envelope  $B(t)$  is also symmetrical, the spectrum of the signal being investigated has an expression

$$\tilde{x}(\omega) = \int_{-T/2}^{+T/2} B(t) e^{j[\varphi(t) - \omega t]} dt = 2 \int_0^{T/2} B(t) \cos[\varphi(t) - \omega t] dt.$$

For calculation  $\tilde{x}(\omega)$  we will use the principle of steady state. We can use formula (8.24), after taking real part from both its parts. As a result it is obtained

$$\tilde{x}(\omega) = 2 \sqrt{\frac{2\pi}{\omega_c(t_0)}} B(t_0) \cos\left[\varphi(t_0) - \omega t_0 + \frac{\pi}{4}\right] + 0(1/m). \quad (8.42)$$

Here, as earlier,  $t_0$  - stationary point, determined by equation (8.26), but now  $t_0$  must be placed in half of the duration:

$$0 < t_0 < T/2.$$

Instantaneous frequency  $\omega(t)$  is assumed to be that increasing in this half of impulse/momentum/pulse (otherwise formula (8.42) is

insignificantly changed). Let us point out also that relationship/ratio (8.42) is unsuitable with  $t_0 \rightarrow 0$ , when rate of change in instantaneous frequency  $\omega_c'(t) = q''(t)$  becomes low.

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For this region is necessary more general method (asymptotic approximation/approach of the third order - see [6]), but here we will not concern this refinement.

Spectrum  $\tilde{X}(\omega)$  is real function, i.e., phase spectrum  $\beta_x(\omega)$  takes only two values - 0 or  $\pi$ . There are here also an analogy with FM signals which possess a similar property, but in the temporary/time, but not in the frequency representation.

Functioning according to the methodology accepted, let us fix arbitrary ChM signal  $x(t)$  from the class in question and is determined the first nearest to it signal of set  $Y$ . According to the theorem of §7.2, for this it is necessary to equate phase spectra  $\alpha(\omega)$  and  $\beta_x(\omega)$ , so that spectrum  $\tilde{Y}(\omega)$  will prove to be real and coinciding in sign  $\tilde{X}(\omega)$ . Then, varying signal  $x(t)$ , we will obtain shortest distance  $d_{min}$  between sets  $X$  and  $Y$ . This consideration completely corresponds to the derivation of formula (8.16), and we come to the maximization of value

$$C(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(\omega) b_x(\omega) d\omega.$$

where in accordance with approximation formula (8.42)

$$b_x(\omega) = 2 \sqrt{\frac{2\pi}{\omega_c(t_0)}} B(t_0) \left| \cos \left[ \varphi(t_0) - \omega t_0 + \frac{\pi}{4} \right] \right| + O(1/m).$$

As a result, passing in the integral to the variable/alternating  $t_0$  [in this case is used equation (8.26)], we obtain the condition of optimum character for the unknown signal in the form, analogous (8.35):

$$C(x, Y) = \sqrt{\frac{2}{\pi}} \int_0^{T/2} B(t) a(\omega_c) \left[ \frac{d\omega_c}{dt} \right]^{1/2} \times \\ \times \left| \cos \left[ \varphi(t) - \omega t + \frac{\pi}{4} \right] \right| dt = \max. \quad (8.43)$$

Here there is a cofactor  $|\cos[\dots]|$ , caused by the more complicated structure of the amplitude spectrum with symmetrical ChM. However, after using the expansion

$$\cos z = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1} \cos 2n\pi z,$$

it is not difficult to note that the main contribution to integral (8.43) introduces first term, not containing rapidly-oscillating factor.

Being limited to this component/term/addend (role of the others let us consider later, in connection with FM signals), we obtain

$$C(X, Y) \approx \left(\frac{2}{\pi}\right)^{3/2} \int_0^{T/2} B(t) a(\omega_c) \left[\frac{d\omega_c}{dt}\right]^{1/2} dt. \quad (8.44)$$

As earlier, for the research for the maximum it is possible to use Schwarz - Buniakowski. Integral (8.44) is maximum, if the factors of the integrand are proportional, i.e.,

$$a(\omega_c) \left[\frac{d\omega_c}{dt}\right]^{1/2} = \gamma B(t). \quad (8.45)$$

By satisfaction of this condition is achieved the shortest distance between X and Y, therefore, from (8.44) is obtained

$$C(X, Y) = \gamma \left(\frac{2}{\pi}\right)^{3/2} \int_0^{T/2} B^2(t) dt = \frac{\gamma}{\pi} \sqrt{\frac{2}{\pi}}. \quad (8.46)$$

In order to determine coefficient  $\gamma$  let us raise equation (8.45) into the square let us integrate from zero to  $T/2$ . Taking into account standardization, we find

$$\gamma^2 \int_0^{T/2} B^2(t) dt = \frac{\gamma^2}{2} = \int_0^{T/2} a^2(\omega_c) \frac{d\omega_c}{dt} dt.$$

Let us switch over in the integral to the right to variable/alternating  $\omega_c$ . This it gives

$$\gamma^2 = 2 \int_{\omega_c(0)}^{\omega_c(T/2)} a^2(\omega_c) d\omega_c.$$

Here integration limits correspond to total variation in the instantaneous frequency of the unknown signal. As usual, we assume that this frequency region contains the range in which is contained

basic part of the energy of the assigned spectrum. Therefore taking into account standardization it is obtained

$$\gamma^2 \approx 2 \int_{-\infty}^{+\infty} a^2(\omega_c) d\omega_c = 4\pi.$$

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As a result formula (8.46) gives<sup>1</sup>

$$C(X, Y) = \frac{2\sqrt{2}}{\pi} + 0(1/\sqrt{m}) = 0.9 + 0(1/\sqrt{m}). \quad (8.47)$$

FOOTNOTE 1. Correction term  $0(1/\sqrt{m})$  considers an error in the previous calculation. We will refine this correction in Chapter 9.

ENDFOOTNOTE.

Simultaneously equation (8.45), which is determining the required law of frequency modulation, obtains the form

$$B^2(t) dt = \frac{1}{4\pi} a^2(\omega_c) d\omega_c. \quad (8.48)$$

It is obvious, we obtained the results, close to previous. Equation (8.48) is similar (to 8.36). Difference in the coefficients is connected with the fact that now total variation in the instantaneous frequency occurs for half of the pulse duration. However, the comparison of the coefficients of proximity (8.38) and (8.47) indicates the essential difference.

It is earlier, for the signals with a monotone change in the

frequency the coefficient of proximity was approximately/exemplarily equal to unity, more precise value  $C(X, Y)$  how conveniently differed little from unity during sufficiently large compression  $m$ . This means that, increasing compression, we could arbitrarily reduce distance  $d_{min}$ , obtaining how conveniently high degree of approximation.

For the signals with symmetrical ChM the coefficient of proximity does not attain one even within the limit with  $m \rightarrow \infty$ . There is a final distance between sets  $X$  and  $Y$

$$d_{min}^2 = 2[1 - C(X, Y)] = 2[1 - 0.9] = 0.2,$$

characterizing the limit of error in the approximation/approach to the assigned amplitude spectrum.

This is clarified in Fig. 8.6, where they are depicted the assigned spectrum  $a(\omega)$  and amplitude spectrum  $b_x(\omega)$  of signal with symmetrical ChM. In accordance with (8.42) the latter has a character of the "rectified sinusoid", since contains factor  $|\cos[...]|$ . Selecting the law of modulation, it is possible to approach the spectra indicated "on the average", as shown in figure, but the available dips/troughs in the points where  $b_x(\omega) = 0$ , limit the quality of approximation/approach.

There is no this limitation for the signals with the monotone law of modulations whose amplitude spectra have steady character and are distorted only due to comparatively small Fresnel pulsations. This difference reflect the obtained values of the coefficients of proximity.

Let us point out also a difference in the procedural nature. For ChM signals with the monotone law of modulation it is possible to arrive at results presented above somewhat simpler. It suffices to require so that the amplitude spectrum  $b_x(\omega)$ , expressed by approximation formula (8.27), would coincide with the assigned spectrum  $a(\omega)$ . This method is used, for example, in [7]. However, from previous it is clear that this consideration is correct only because without the account to an error in the asymptotic approximations/approaches it is possible to obtain a precise conformity between the given one and that unknown by the spectra (coefficient of proximity  $C(X, Y)$  it is equal to unity). For the signals with symmetrical ChM this not thus. We will not obtain the solution, if we will attempt to equate assigned and unknown the spectra. Here it is possible to only carry out approximation/approach of these spectra with final error. Our approach, based on the criterion of proximity, contains both cases, while the utilized previously methods are suitable only for the first task.

The task examined about the signals with symmetrical ChM is of special interest also because it has very close analogy with the synthesis of signals with the phase manipulation. This question is in detail examined further.

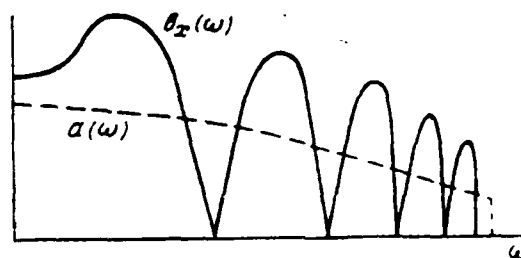


Fig. 8.6.

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Chapter 9.

## SYNTHESIS OF FM SIGNALS.

### 9.1. Quantized and not quantized FM signals.

Signals with the phase manipulation are cscillations with constant amplitude and constant frequency of filling, whose initial carrier frequency is changed with jumps at some moments/torques  $t_1, t_2, \dots, t_n, \dots$  and it can take fixed values  $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$

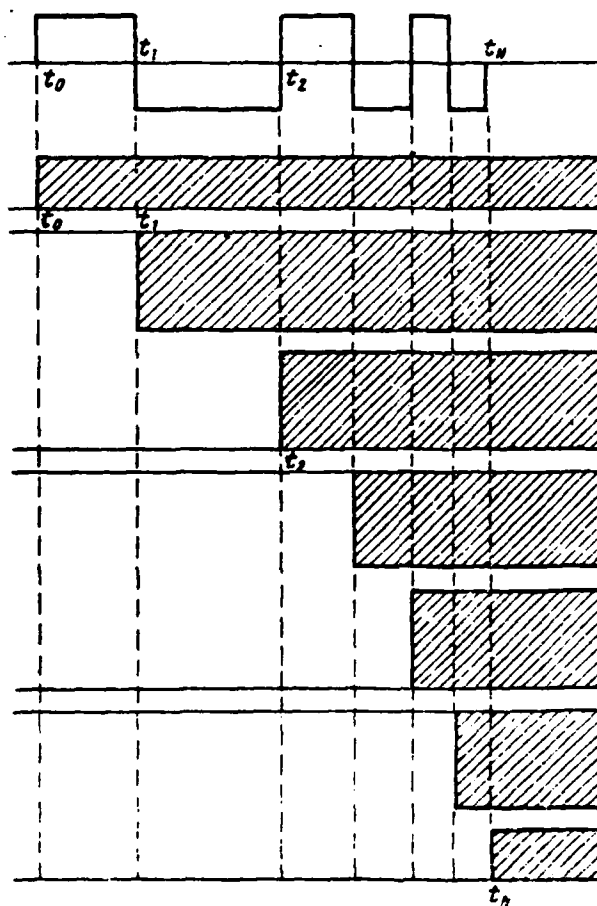


Fig. 9.1.

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A fundamental use find binary FM signals, which allow/assume only the two values of initial phase  $\phi_1=0$  and  $\phi_2=\pi$ . Only such signals are examined below. Composite envelope of binary FM signal is step function - square wave - with the commutations of sign at

moments/torques  $t_h$  (Fig. 9.1). The key advantage of such signals over the signals of another type lies in the fact that is not required a steady change in the parameter; the intermittent character of modulation of phase (manipulation) makes it possible to ensure necessary accuracy with a comparatively simple equipment.

Let us give the determination of binary FM signal. Let the function  $\lambda(t)$  be equal to  $\pm 1$  for all values of  $t$ , except certain multitude of values  $t_h$  of zero measure; at the points of last set the function  $\lambda(t)$  endures intermittent sign changes. Functions  $\lambda(t)$  are the envelopes of FM signals of single amplitude and infinite duration. FM signals of final duration are obtained from  $\lambda(t)$  by limitation in time. Furthermore, we normalize signal amplitude so that its energy would be equal to unity. Finally we have

$$x(t) = \begin{cases} 1/\sqrt{T} \lambda(t) & \text{при } -T/2 \leq t \leq T/2; \\ 0 & \text{при } |t| > T/2. \end{cases} \quad (9.1)$$

Key: (1). with.

From Fig. 9.1 it is clear that FM signal it is possible to depict also as the imposition of the functions of inclusion/connection. The amplitudes of jumps have single value in the beginning and at the end of the signal and the doubled value - at intermediate points. Therefore, after designating total number of jumps through  $N$ , we obtain another form of recording, equivalent (9.1):

$$x(t) = \frac{2}{\sqrt{T}} \left\{ \frac{1(t-t_0)}{2} + \sum_{k=1}^{N-1} (-1)^k 1(t-t_k) + \frac{(-1)^N}{2} 1(t-t_N) \right\}. \quad (9.2)$$

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Since the spectrum of function  $1(t)$  is  $1/j\omega$ , for the spectrum of FM signal it is easy to obtain expression<sup>1</sup>

$$\tilde{x}(\omega) = \frac{2}{j\omega \sqrt{T}} \left\{ \frac{e^{-j\omega t_0}}{2} + \sum_{k=1}^{N-1} (-1)^k e^{-j\omega t_k} + (-1)^N \frac{e^{-j\omega t_N}}{2} \right\} \quad (9.3)$$

FOOTNOTE 1. Here is not taken into consideration deltoid component in the spectrum of the function of the inclusion, it does not play in this case any role. ENDFCCTNCTE.

Here  $t_k$  the moments/torques of the commutation of phase,  $t_0 = -T/2$ ,  $t_N = T/2$ .

It was above assumed that the moments/torques of commutation can be placed arbitrarily on duration  $T$ . However, fundamental use/application find FM signals, comprised of the samples of fixed period of time. Not to the detriment of the generality we will

further assume/set this duration of single. The moments/torques of commutation for such signals are multiple the duration indicated, i.e.,  $t_k = v_k$  - whole numbers. We will call these signals those quantified (KFM), keeping in mind quantization on the time.

It is not difficult to give the determination of KFM signal, analogous (9.1). Let numbers  $\lambda_i$  take values of  $\pm 1$ . Then KFM signal, which consists of the  $n$  samples, can be presented in the form

$$x(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \lambda_i u_0(t-i) = \sum_{i=1}^n x_i u_0(t-i). \quad (9.4)$$

Here, as earlier, factor  $1/\sqrt{n}$  provides standardization on the energy, since under the stipulated conditions the duration of signal  $T$  is equal to a number of samples  $n$ ;  $u_0(t)$  is a square pulse of the single duration

$$u_0(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t > 1 \end{cases}$$

Key: (1). with.

value  $x_i = \lambda_i / \sqrt{n}$  - the amplitude of samples.

From (9.4) it is clear that KFM signals relate to the composite/compound, examined in §7.4.

The spectrum of KFM signal can be presented in the form <sup>see</sup> (7.31)

$$\tilde{x}(\omega) = \tilde{u}_0(\omega) H(\omega), \quad (9.5)$$

where

$$\tilde{u}_0(\omega) = \frac{\sin \omega/2}{\omega/2} \quad (9.6)$$

- spectrum sample, and

$$H(\omega) = \sum_{i=1}^n x_i e^{-j\omega i} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \lambda_i e^{-j\omega i} \quad (9.7)$$

- spectrum of the code.

Formulas (9.4) - (9.7) do not contradict previous, since KFM signals belong to the total set of FM signals with the arbitrary arrangement/position of commutations. The isolation/liberation of KFM signals into the independent class is explained, mainly, by the fact that during their formation and processing it is to use elements/cells of digital computer technology - shift registers, the pulse counters, etc. This it simplifies to apparatus. Moreover, by the methods of the synthesis of KFM signals are characteristic some special features/peculiarities, which consider their discrete/digital time structure.

## 9.2. Brief survey/coverage of best KFM signals.

It is obvious, that KFM signal, which contains  $n$  of samples is completely determined by the sequence of coefficients  $\lambda_i$  equal to  $\pm 1$ ,

or -1. Therefore to each such signal it is possible to supply in the conformity the binary numerical sequence, which is determining its structure. For the analysis of the sequences indicated are applied the methods of the theory of numbers, the algebra of binary polynomials, combinational analysis or another discrete/digital apparatus. These methods are the basis of the overwhelming majority of works on the synthesis of FM signals. Although the precisely discrete/digital methods made it possible to obtain the majority of known KFM signals with good properties, to us it is represented by that not substantiated to be limited only to such methods of synthesis. Keeping in mind to clarify the aforesaid, let us consider briefly fundamental KFM signals.

Special position occupy Barker's signals, proposed in 1953 g [3]. These signals have smallest possible remainders/residues of the autocorrelation function, which do not exceed  $1/n$ . It is possible to show [7] that the spectrum of code  $H(\omega)$  for Barker's signals least deviates on modulus (in the sense of quadratic approximation/approach) from unity. Consequently, according to presented in §7.4, Barker's signals provide best square approximation to the spectrum of single sample - square impulse/momentum/pulse.

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By Barker were indicated the codes with the remainders/residues  $1/n$  only for  $n \leq 13$ . These codes are obtained by the selection: after the calculation of corresponding correlation functions were selected/taken those signals, for which the remainders/residues do not exceed  $1/n$ . Were done the repeated attempts to find Barker codes for  $n > 13$ . In particular, are indications that were tried all binary sequences for  $n \leq 31$  [4] <sup>1</sup>.

FOOTNOTE <sup>1</sup>. Let us note that only with  $n=31$  there is  $2^{31} \approx 2 \cdot 10^9$  different signals. ENDFOOTNOTE.

However, after research of a number of the authors it is possible to consider it established/installed that Barker codes for  $n > 13$  there does not exist [32, 77].

Because of the need to apply KFM signals with a large number of samples, were revealed some other codes with the remainders/residues, large than  $1/n$ , but by nevertheless sufficiently low. Here should be

noted M-sequences and signals, which use the given residue classes. The particular form of the latter are the codes of Legendre's symbols, called also Pell -Plotkin's codes.

The synthesis of the signals indicated is reduced to the following. First, is applied certain algorithm (selected a priori, without the proof of its optimum character) for the construction of the infinite periodic sequence of binary symbols (+1 and -1 or 0 and 1).

In the case of M-sequences is assigned one of the irreducible binary polynomials of the corresponding degree  $q$ . There is a comparatively complete table of such polynomials [51], and sequence is constructed according to the coefficients of polynomial by completely regular form [13, 84] <sup>2</sup>.

FOOTNOTE <sup>2</sup>. Are known also other equivalent algorithms, which lead to M-sequences [44, 81]. ENDFOOTNOTE.

The period of sequence comprises

$$n=2^q-1.$$

where  $q$  - whole number.

For each  $q$  are several irreducible polynomials. Thus, for  $q=6$

(number of signs in period  $n=63$ ) it is known 6 irreducible polynomials, for  $q=7$  (number of signs  $n=127$ ) - 18 polynomials. Choosing one or another polynomial, it is possible to obtain signal with one or the other properties, and there is no general rule for this selection <sup>3</sup>.

FOOTNOTE <sup>3</sup>. With the selection of polynomial frequently is considered the larger or smaller complexity of signal shaper [81], but this question here is not examined. ENDFOOTNOTE.

Signal in the form of periodic infinite sequence usually is not applied in the radar, it is necessary to still select the appropriate segment of this sequence, and here also is certain arbitrariness (see below).

For the construction of the sequence of the given residue classes is used the algorithm of another kind [4]. Here the period of sequence  $n$  is assigned equal to the prime number  $P$  and is chosen one of the primitive roots of this number  $g$ . The property of primitive roots consists in the fact that, raising root of  $g$  to the degree from 0 to  $P-1$ , we obtain the numbers whose deductions, undertaken on modulus/module  $P$ , take all values from 0 to  $P-1$  inclusively. These deductions can be decomposed into some given residue classes. Further to one group of residue classes is assigned symbol  $+1$ , and to

remaining classes - symbol - 1.

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Finally, computing consecutive indices of a radical  $g$ , is constructed sequence from  $+1$  and  $-1$ , sign in this case is chosen in accordance with the fact, to which residue class belongs the next degree of primitive root.

Although the rules indicated assume the sorting of a sufficiently large number of versions, they provide the synthesis of good KFM signals with the permissible space of calculations. It is established/installed, that the sequences indicated possess the minimum remainders/residues of autocorrelation function. But here have in mind the autocorrelation functions of infinite periodic sequence.

In order to use the sequences indicated in the radar (and also in some communicating systems), it is necessary to be bounded to the segment of finite length, which leads to a further deterioration in the correlation properties.

It is proved that the autocorrelation function of one period of sequence (aperiodic autocorrelation function) is connected with the

correlation properties of infinite sequences. The low remainders/residues of the autocorrelation function of aperiodic signal can be obtained only when the remainders/residues of the autocorrelation function of the infinite sequence, formed by its repetition, are also low [4]. Therefore, synthesizing aperiodic signal, it is expedient to use one period of a good periodic sequence of the number of those indicated above. However, beginning signal from different elements/cells of sequence, we will obtain noncyclic signals with the worse or best properties. In other words, the cyclic permutation of the elements/cells of sequence in the limits of one period leads to the different value of remainders/residues. Usually it is necessary to sort out all  $n$  of signals and to select/take the best of them on the level of remainders/residues.

Employing high speed calculating means, it is possible to surmount the appearing difficulties. During the latter/last decade were obtained much KFM signals with good correlation functions. In Table 9.1 are generalized the results, indicated in works [4, 45, 50], moreover are here given signals with the smallest known level of remainders/residues. In the table are accepted the following designations. Signals are assumed to be those not standardized, so that the main peak of correlation function is equal to a number of samples  $n$ . The greatest remainder/residue is expressed by whole number  $\mu$ . In the previous designations the level of the greatest

remainder/residue comprises  $p/n$ . In the fourth column of table is indicated the type of signal, in this case the letter of "M" designates M-sequence, "E" - sequence of the given residue classes (among other things of Legendre's symbols), letter "Dr" mean that the signal does not relate to the types indicated.

It is known that the level of maximum remainder/residue for KFM signals is close to value  $1/\sqrt{n}$  (besides Barker's signals). This is illustrated by data in the third column Table 9.1. Is here indicated value  $k$ , equal to the relation of the maximum remainder/residue to  $\sqrt{n}$ . For the normalized autocorrelation function (principal maximum of which is equal to unity) we have

$$R_{max} = k/\sqrt{n}. \quad (9.8)$$

In the region in question we obtained

$$k \approx 0.6 + 0.8.$$

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Table 9.1.

n	μ	k	(1) Тип	n	μ	k	Тип	n	μ	k	Тип
15	3	0,77	M	133	7	0,66	B	383	15	0,77	B
16	2	0,5	Др	127	7	0,62	B	397	16	0,80	B
17	2	0,49	B	127	8	0,71	M	401	15	0,75	B
19	3	0,69	B	139	8	0,68	B	409	15	0,74	B
23	3	0,63	B	151	8	0,65	B	419	15	0,73	B
28	2	0,38	Др	157	8	0,64	B	431	15	0,72	B
29	3	0,56	Др	163	9	0,70	B	443	15	0,71	B
29	4	0,75	B	167	8	0,62	B	449	15	0,71	B
31	3	0,53	B	179	9	0,69	B	467	15	0,70	B
31	4	0,71	M	191	9	0,65	B	487	15	0,68	B
37	4	0,66	B	193	10	0,72	B	491	16	0,72	B
41	4	0,63	B	199	10	0,71	B	499	17	0,76	B
43	4	0,61	B	211	10	0,69	B	503	18	0,80	B
47	4	0,59	B	223	10	0,67	B	521	17	0,75	B
53	5	0,69	B	227	10	0,65	B	547	18	0,77	B
59	5	0,65	B	233	11	0,72	B	563	18	0,76	B
61	5	0,64	B	239	12	0,78	B	577	17	0,71	B
63	6	0,76	M	251	11	0,70	B	587	19	0,78	B
67	5	0,61	B	255	13	0,81	M	599	19	0,78	B
71	5	0,59	B	257	12	0,75	B	607	19	0,77	B
73	6	0,70	B	283	12	0,72	B	619	19	0,76	B
79	6	0,68	B	293	13	0,76	B	631	18	0,72	B
83	6	0,66	B	311	13	0,74	B	643	19	0,75	B
89	6	0,64	B	317	12	0,68	B	653	20	0,78	B
97	7	0,71	B	321	14	0,77	B	659	19	0,74	B
101	6	0,60	B	347	14	0,75	B	661	20	0,78	B
103	8	0,70	B	353	15	0,80	B	674	21	0,81	B
107	7	0,68	B	359	14	0,74	B	683	20	0,77	B
109	8	0,77	B	379	14	0,72	B	709	20	0,77	B

Key: (1). Type.

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In other words, for the signals with large  $n$  the maximum level of remainders/residues is close to  $1/\sqrt{n}$ .

Let us emphasize again that are here used the best achievements

of latter/last time. In particular, from the data of Table 9.1 it is evident that M-sequences are not optimum: value  $k$  for them somewhat higher than for other signals.

### 9.3. On the methods of synthesis FM of signals.

One should assume that many authors' intense searches to a considerable degree drained the possibility of the synthesis of KFM signals. It is difficult to expect from other methods of substantially best results. But approach itself to the synthesis causes, in our opinion, certain dissatisfaction.

In fact, for FM signals there are no regular, variational methods of synthesis. Different methods use different a priori algorithms for the construction of infinite sequences. Furthermore, necessary is selection during the determination of the best cyclic permutation, which gives the minimum of remainders/residues. With this approach one cannot, of course, be confident that was obtained the signal with the best (in the assigned sense) properties.

With a large number of samples the sorting versions proves to be very bulky and can prove to be problem even for contemporary TsVM. Meanwhile there is a clear tendency to apply FM signals with all by a large number of samples, and this justifies the search for other

methods of synthesis, not connected with similar difficulties.

Focuses attention also a qualitative difference in the methods of synthesis of ChM and FM signals. As we saw, synthesis of ChM signals is produced by regular methods, without the selection, and asymptotic decision provides the necessary accuracy precisely during the large compression, i.e., in that region where the synthesis of FM signals is most hindered/hampered. The methods of synthesis of ChM signals possess physical clarity. There is a sufficiently obvious connection/communication between instantaneous speed of modulation and level of the spectrum, that makes it possible to come to light/detect/expose the structure of the unknown law ChM according to the assigned autocorrelation function. The known methods of synthesis of FM signals completely disregard similar physical considerations.

We will show further that the criterion of proximity permits to work out substantially another method of synthesis - method, which does not require selection and which reveals/detects generality, the inherent in ChM and FM for oscillations. The proposed below asymptotic decision allows/assures physical interpretation and it is useful for FM signals with the large compression. Although during the use of this method are not obtained the best signals, than given higher, it has advantages in the sense of simplicity and clarity.

As it follows from that presented, with the synthesis of KFM signals it is accepted to use a minimax criterion of the approximation/approach: optimum considers signal the smallest level of the greatest remainder/residue. This criterion answers the essence of problem. Remainders/residues (minor lobes) mask signals from the weak close targets, and it is desirable to bound the level of remainders/residues by the permissible low value. We will, however, apply the hypothesis of proximity in space  $L^2$ , which corresponds to the quadratic approximations/approaches, which characterize somehow the average/mean, but not maximum level of remainders/residues.

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Of course this is connected, first of all, with the fact that the quadratic criterion simplifies decision, but there are other considerations in favor of this criterion.

The level of the greatest remainder/residue unconditionally characterizes resolution, if discussion deals with resolution of two targets. Comparing signal from one target with the maximum remainder/residue from another target, we obtain the evaluation of permission/resolution under the worst conditions. But, if is traced resolution with the multiple chaotically arranged/located mixing reflections when signal at each moment of time is formed during the

imposition of many elementary responses, position substantially varies. Under these conditions the task of permission/resolution has a static character and a quality of signal it is characterized by the root-mean-square level of remainders/residues, but minimax. This will be coordinated with the criterion, utilized below, see also [15].

In this chapter further is examined the synthesis of the not quantized by FM signals with the arbitrary arrangement/position of commutations. Such signals form wider set than KFM, but on the set in question are retained the fundamental special features/peculiarities of phase manipulation - intermittent character and the constancy of amplitude. These special features/peculiarities characterize main technical advantages of FM signals over the signals of other types and, as it is clear from the following, precisely, they are determining in the problem of synthesis. We will show that the important properties of FM signals which, until now, could be only assumed on the base of available experiment, are revealed/detected completely naturally with the help of the proposed method. In particular, asymptotic decision will come to light/detect/expose the maximum level of remainders/residues, which is approximately coordinated during the large compression with the given results.

But nevertheless larger practical interest represent KFM signals. The corresponding methods of synthesis are examined in the

following chapter. These methods are the straight/direct development of the methods, set forth below.

#### 9.4. Approximations/approaches on the set FM of signals.

Let us switch over to synthesis of FM signals. As we usually assume that in the space of signals  $H$  are many  $X$  permissible signals and many  $Y$  desired signals. Set  $X$  contains all FM signals with the arbitrary arrangement/position of commutations, i.e., satisfying conditions (9.1) - (9.3). The structure of set  $Y$  depends on specific problem. If synthesis of FM signal is produced according to the function of uncertainty/indeterminacy  $\chi(t, \Omega)$ , by realizable certain, in general, by the continuous signal  $s(t)$ , then set  $Y$  contains all signals, which possess this function of uncertainty/indeterminacy. As it was shown in chapter 7, these signals are characterized by only initial phase, i.e.

$$y_i(t) = s(t) e^{j\phi_i}. \quad (9.9)$$

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We will trace also synthesis of FM signals according to the assigned realizable autocorrelation function  $R(t)$ . In this case set  $Y$  contains all signals with assigned  $R(t)$ , i.e., with the assigned amplitude spectrum  $a(\omega)$ , connected in a known manner with  $R(t)$ ,

$$\tilde{y}(\omega) = a(\omega) e^{-j\phi(\omega)}. \quad (9.10)$$

where  $\alpha(\omega)$  - arbitrary phase spectrum.

Let us designate through  $X_T$  the set of FM signals of the assigned duration  $T$ . Let there be the arbitrary signal  $y(t)$ . Let us find FM signal  $x(t)$ , which belongs to set  $X_T$ , ensuring best approximation to  $y(t)$ , i.e., will solve the task of approximation on the set of FM signals. Result gives the following theorem.

a) Best approximation to signal  $y(t)$  gives on set  $X_T$  signal  $x(t)$ , expressed by formula (9.1), for which the function  $\lambda(t)$  is determined with  $-T/2 < t < T/2$  by the condition

$$\lambda(t) = \begin{cases} +1 & \text{для всех } t, \text{ при которых } \operatorname{Re} y(t) > 0; \\ 0 & \\ -1 & \text{для всех } t, \text{ при которых } \operatorname{Re} y(t) < 0, \end{cases} \quad (9.11)$$

Key: (1). for all  $t$  with which.

or otherwise  $\lambda(t) = \operatorname{sign} \operatorname{Re} y(t)$ , where  $\operatorname{sign}$  - function of sign. The moments/torques of commutation of signal  $x(t)$  coincide with zero  $\operatorname{Re} y(t)$ , i.e.

$$\operatorname{Re} y(t_k) = 0$$

b) If with  $-T/2 < t < T/2$  real part  $\operatorname{Re} y(t)$  is different from zero in any interval of  $t$  of final measure, the signal of the best approximation on set  $X_T$  only.

c) The coefficient of proximity between signal  $y(t)$  and set  $X_T$  comprises

$$C(X_T, y) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} \operatorname{Re} y(t) dt. \quad (9.13)$$

Actually/really, as usual, the task of approximation is reduced to the maximization of the coefficient of proximity which taking into account (9.1) obtains the expression

$$C(x, y) = \operatorname{Re} \int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} \lambda(t) \operatorname{Re} y(t) dt.$$

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According to the condition  $\lambda(t) = \pm 1$ , therefore,

$$C(x, y) \leq \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} |\lambda(t) \operatorname{Re} y(t)| dt = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} |\operatorname{Re} y(t)| dt.$$

Is here achieved equality only by satisfaction of condition (9.11) which, as can easily be seen, proves all confirmations of theorem.

For the uniqueness of the best approximation it is significant that function  $\operatorname{Re} y(t)$  is different from zero in any finite interval with  $-T/2 < t < T/2$ . Actually/really, if in certain section  $\tau \subset T$  function  $\operatorname{Re} y(t) \equiv 0$ , then, as it is clear from the formula for the coefficient of proximity, its value does not depend on what function  $\lambda(t)$  is selected in this section. Consequently, under these conditions best approximation is not unambiguous.

The proved theorem has fundamental value for the synthesis of FM signals. It determines best approximation on set  $X_T$  for any continuous signal  $y(t)$ . The fundamental content of theorem is reduced to the very simple rule: for obtaining the best approximation on set of FM signals it is necessary and it suffices to produce the ideal limitation of the assigned signal (are more precise, its real parts), so as to the given one and approximating signals would coincide in the sign (Fig. 9.2).

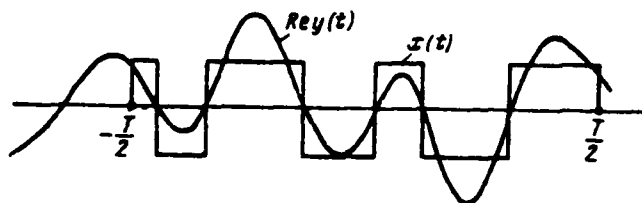


Fig. 9.2.

#### 9.5. Synthesis according to the function of uncertainty/indeterminacy.

Let us use the proved in the previous paragraph theorem for the synthesis of FM signal according to the realizable function of uncertainty/indeterminacy. As it was noted, set  $Y$  contains in this case the signals, which are characterized by only initial phase and determined by relationship/ratio (9.9).

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In connection with this task we confirmed the hypothesis of proximity in chapter 8, after showing that best quadratic approximation to the assigned function of uncertainty/indeterminacy gives signal  $x_{opt}(t)$ , nearest to the set  $Y$  indicated.

For finding this signal let us use the following order of the minimization of distance. First, fixing/recording arbitrary signal

$y \in Y$ , it is determined the nearest to it signal of set  $X$ . If the latter contains FM signals of prescribed duration  $T$ , then result is determined by the previous theorem and distance  $d(X, y)$  is characterized by the coefficient of proximity (9.13). It is possible to consider, however, the more general case when duration of FM signal  $T$  is not assigned previously, but it must be determined in the process of synthesis. Then it is necessary to maximize the coefficient of proximity also in value  $T$ , i.e., taking into account (9.13):

$$C(X, y) = \max_r C(X_r, y) = \max_r \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} |\text{Re}s(t) e^{j\phi_0}| dt.$$

Then we will maximize the coefficient of proximity also in set  $Y$ . In the task in question the signals of set  $Y$  satisfy condition (9.9) and differ from each other only in terms of initial phase  $\phi_0$ . Therefore

$$C(X, Y) = \max_{y \in Y} C(X, y) = \max_{\phi_0} \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} |\text{Re}s(t) e^{j\phi_0}| dt.$$

Separating/liberating in signal  $s(t)$  amplitude and phase factors  $s(t) = A(t) e^{j\phi(t)}$ , we finally obtain

$$C(X, Y) = \max_{\phi_0} \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} |A(t)| |\cos[\Phi(t) - \phi_0]| dt. \quad (9.14)$$

Let us consider a specific example. As it was shown in chapter 6, optimum function uncertainties/indeterminacies in the sense of its concentration in certain central circle give the function of

Hermite. We will seek approximation/approach to this function of uncertainty/indeterminacy of FM signal.

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The function of zero-order Hermite - Gaussian signal - has a compression of the order of one. Attempting to obtain FM signal with the large compression, logical as the "sample/specimen" to take the function of Hermite of higher order. Let us place

$$s(t) = A(t) = (n! 2^n / \pi)^{-1/2} e^{-t^2/2} H_n(t), \quad (9.15)$$

where  $H_n$  - Hermite's polynomial.

We examine the real signal for which  $\Phi(t) \equiv 0$ . Therefore from (9.14) it is obtained

$$C(X, Y) = \max_{\tau, \phi_0} \frac{|\cos \phi_0|}{V T} \int_{-T/2}^{T/2} |s(t)| dt.$$

Obviously, maximum on  $\phi_0$  occurs with  $\phi_0 = 0$ . This condition determines signal  $y_{opt}$ , nearest to set  $X$ . Consequently,

$$C(X, Y) = \max_{\tau} \frac{1}{V T} \int_{-T/2}^{T/2} |s(t)| dt. \quad (9.16)$$

Fig. 9.3 show the assigned signal, the function of Hermite of the 10th order. Stepped line represents unknown envelope of FM signal, constructed in accordance with the previous theorem (i.e. so that the moments/torque of commutation fall to zero  $s(\tau)$ ). The optimum duration  $T$  is determined according to condition (9.16). Broken line showed the dependence of integral (9.16) of the duration.

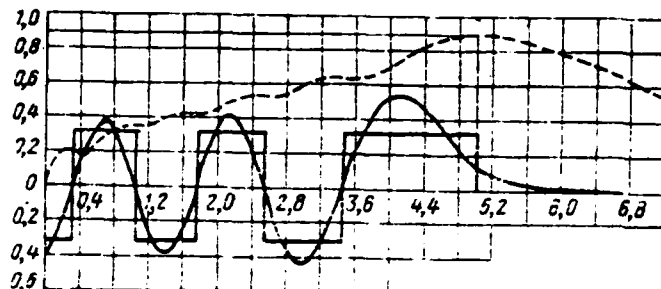


Fig. 9.3.

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Maximum occurs with  $T/2 \approx 5$ , that also determines the unknown duration. In this case

$$C_{\max} = C(X, Y) \approx 0.87.$$

As usual, the value of the maximum coefficient of proximity determines the shortest distance between sets  $X$  and  $Y$ :

$$d_{\min}^2 = 2[1 - C(X, Y)] = 2(1 - 0.87) = 0.26.$$

In § 7.1 it was shown that the smallest quadratic difference in the corresponding functions of uncertainty/indeterminacy also is expressed as the coefficient of the proximity:

$$\begin{aligned} d_{\min}^2(Z_X, Z_Y) &= \min_{X \in Y} \frac{1}{2\pi} \int_{-\infty}^{\infty} |Z_X(t, \Omega) - Z_Y(t, \Omega)|^2 d\Omega dt = \\ &= 2[1 - C^2(X, Y)] = 2(1 - 0.87^2) = 0.49 \end{aligned}$$

The comparison of the two-dimensional functions of uncertainty/indeterminacy, represented in the form of the surfaces above the plane  $(t, \Omega)$ , presents known difficulty. We will be bounded to two sections, shown in Fig. 9.4. The function of the uncertainty/indeterminacy of hermitian signal is a body of revolution.

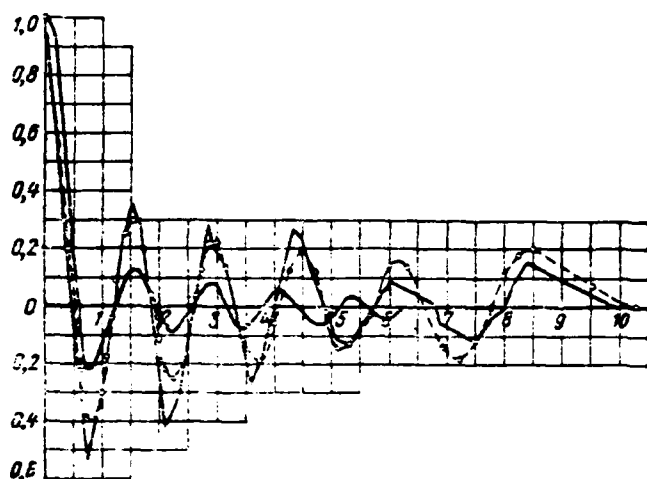


Fig. 9.4.

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Its section along the time axis (autocorrelation function) has an expression

$$\chi_s(t, 0) = R(t) = e^{-t^2/4} L_n(t^2/2),$$

where  $L_n$  - Laguerre's polynomial.

Analogously occurs section along the axis of the frequencies

$$\chi_s(0, \Omega) = e^{-\Omega^2/4} L_n(\Omega^2/2).$$

This dependence is shown in Fig. 9.4 by dotted line. Solid lines represent the appropriate sections of the function of uncertainty/indeterminacy  $\chi_s(t, \Omega)$  for the approximating FM signal, shown in Fig. 9.3.

We obtained satisfactory approximation/approach. For the autocorrelation function, which depends on the code, broken line, which corresponds to FM signal sufficiently fully describes even fine structure of the function of laguerre, approximately repeating the form of main surge and all remainders/residues. Sections  $\chi(0, \Omega)$  approach to a lesser degree. Here satisfactory coincidence takes place for the main surge and near lateral ones. Further curves diverge. This is explained by the fact that for FM signal with rectangular envelope section  $\chi(0, \Omega)$  it does not depend on the code, it is described by the function

$$\chi_x(0, \Omega) = \frac{\sin \Omega T/2}{\Omega T/2},$$

which is completely determined by one parameter T. Logically, the possibilities of approaching this section are very limited.

Let us note, however, that the dotted curve, which corresponds in Fig. 9.4 to continuous hermitian signal, is placed in the considerable section between the solid lines. This makes it possible to assume that the approximation/approach of the same order give all other sections. In any case, we obtained best quadratic approximation to the assigned function of uncertainty/indeterminacy.

One should emphasize that the quality of approximation/approach

depends substantially on what function of uncertainty/indeterminacy is assigned. We considered the case when the realizing signal  $s(t)$  is real. This sets known limitation on the structure of the function of uncertainty/indeterminacy <sup>1</sup>.

FOOTNOTE <sup>1</sup>. The latter possesses symmetry relative to the axes of coordinates  $t$  and  $\Omega$ . ENDFCOTNOTE.

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It is obvious, the real generating signal to approach with the help of FM signal (which also real) is possible better than the signal of general view, which has imaginary component. In particular, an attempt at the approximation/approach to ChM signal (in which real imaginary the parts are commensurated on the energy) gives considerably worse results. Respectively, is obtained worse approximation/approach of the functions of uncertainty/indeterminacy [8].

#### 9.6. synthesis according to the autocorrelation function.

We pass to the complex problem - synthesis of FM signals according to the autocorrelation function. As it is clear from the previous survey/coverage, precisely, this problem is the basis of the

majority of the known methods of synthesis FM.

The general/common/total treatment of a question remains previous. If  $R(t)$  - the assigned realizable autocorrelation function, then many desired signals  $Y$  include the signals, which satisfy condition (9.10) and which are characterized by only phase spectrum: the amplitude spectrum is uniquely determined by the assigned autocorrelation function:

$$|\tilde{y}(\omega)|^2 = a^2(\omega) = \int_{-\infty}^{\infty} R(t) e^{-j\omega t} dt.$$

After selecting certain desired signal  $y \in Y$ , we we can determine FM signal  $x \in X$ , nearest to it. In accordance with the theorem § 9.4 for this it is necessary and it suffices to form FM signal  $x(t)$  so that the given one and approximating signals would coincide in the sign. The corresponding coefficient of proximity gives relationship/ratio (9.13) <sup>1</sup>

$$C(X, y) = \frac{1}{T} \int_{-T/2}^{T/2} \text{Re} \tilde{y}(t) dt. \quad (9.13)$$

FOOTNOTE <sup>1</sup>. Here and throughout is examined set of FM signals of fixed period of time  $T$ . This does not lead to the loss of generality, if compression is sufficiently great (see below). ENDFOOTNOTE.

In order to find signal  $y$  nearest to set  $X$ , it is necessary, varying signal  $y$  to obtain the maximum of this value.

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Satisfying this condition signal  $y_{opt}$  is generating for unknown by FM signal  $x_{opt}$  in order to determine the latter, it suffices to further again use theorem of § 9.4.

Thus, the criterion of proximity leads to the following task:

it is necessary to find phase spectrum  $\alpha_{opt}(\omega)$ , the maximizing coefficient of proximity (9.17) when signal  $y(t)$  is connected with  $\alpha(\omega)$  the relationship/ratio

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) e^{-j[\alpha(\omega) - \omega t]} d\omega, \quad (9.18)$$

where  $a(\omega)$  - the assigned amplitude spectrum. The moments/torques of commutation  $t_k$  of the unknown FM signal  $x(t)$  are determined further from the condition

$$\text{Re} y_{opt}(t_k) = 0. \quad (9.19)$$

Signal  $x_{opt}$  satisfies the criterion of proximity, i.e., realizes the minimum of distance of set  $\mathcal{Y}$ . After using another order of the minimization of distance, we demonstrated in § 7.2 that this signal provides best quadratic approximation to the assigned amplitude spectrum  $a(\omega)$  (see formula (7.17)), and it also gives the autocorrelation function, close to the optimum (in the sense of

quadratic approximation/approach - see formulas (7.20), (7.25)).

The proposed method of synthesis is reduced to finding of generating signal  $y_{opt}(t)$ , and not directly not unknown FM signal  $x_{opt}(t)$ . This leads to the variational problem, where unknown is continuous function - phase spectrum  $a(\omega)$ . Failure of the direct synthesis discrete/digital FM of signals allows, as we will see, to work out the regular method which, at least, in the asymptotic approximation/approach gives decision without any selection.

As for ChM signals, is here possible the iterative procedure of decision by the method of successive design. Iterations make it possible to obtain more exact solution, being transmitted from certain signal of zero approximation. Such iterations completely correspond to the overall diagram, presented in § 1.8 and which was being repeatedly applied by us earlier.

After assigning FM signal of the zero approximation  $x_0$ , we determine signal  $y_1$ , which belongs to set  $Y$ , nearest to  $x_0$ .

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We then seek signal  $x_1$  nearest to  $y_1$ . Repeating this process, we obtain descending sequence of the distances

$$d_1 \geq d_2 \geq d_3 \geq \dots$$

Let us dismantle/select in greater detail than these iterations. After assigning FM signal of the zero approximation  $x_0$ , we can determine its spectrum, using relationship/ratio (9.3). If we separate/liberate the real and imaginary parts of the spectrum, it is obtained

$$\begin{aligned}\tilde{x}(\omega) &= \frac{2}{\omega T} \{U(\omega) + jV(\omega)\}, \\ U(\omega) &= \sum_{k=1}^{N-1} (-1)^k \sin \omega t_k^{(0)} + \frac{1}{2} [1 + (-1)^N] \sin \frac{\omega T}{2}, \quad (9.20) \\ V(\omega) &= \sum_{k=1}^{N-1} (-1)^k \cos \omega t_k^{(0)} - \frac{1}{2} [1 + (-1)^N] \cos \frac{\omega T}{2}.\end{aligned}$$

Here  $t_k^{(0)}$  - moments/torques of commutation for signal  $x_0$ . In accordance with the theorem of § 7.2 in order to determine signal  $y_1 \in Y$ , nearest to  $x_0$ , it is necessary for the assigned amplitude spectrum  $a(\omega)$  to ascribe the phase spectrum of signal  $x_0$ . In other words, spectrum  $y(\omega)$  should be registered in the form

$$\begin{aligned}\tilde{y}(\omega) &= a(\omega) \frac{\tilde{x}_0(\omega)}{|\tilde{x}_0(\omega)|} = \\ &= a(\omega) \frac{U(\omega) + jV(\omega)}{\sqrt{U^2(\omega) + V^2(\omega)}} \operatorname{sign} \omega.\end{aligned}$$

Signal  $y_1(\tau)$  as the function of time is determined further by inverse transformation of Fourier:

$$\begin{aligned}y_1(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{y}(\omega) e^{j\omega t} d\omega = \\ &= \frac{1}{\pi} \int_0^{\infty} a(\omega) \frac{U(\omega) \cos \omega t + V(\omega) \sin \omega t}{\sqrt{U^2(\omega) + V^2(\omega)}} d\omega \quad (9.21)\end{aligned}$$

It is here assumed that the amplitude spectrum  $a(\omega)$  occupies the final band  $(-\Omega, \Omega)$  and is even function.

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Finally, it is necessary to fulfill transition from signal  $y_1$  to the nearest signal  $x_1$  of set  $X$ . This transition gives theorem of § 9.4: the moments/torques of commutation of FM signal  $x_1$  correspond to zero real parts of the approximated signal  $y_1(t)$ . Since signal  $y_1(t)$  real, finally is obtained the equation

$$y_1(t_k^{(1)}) = 0; \quad -T/2 < t_k^{(1)} < T/2 \quad (9.22)$$

for determining the moments/torques of commutation of FM signal of first approximation. Further stages of iterations are produced analogously.

In chapter 1 it was shown that this iterative process corresponds to projective-gradient method. Is minimized here distance between  $X$  and  $Y$ . Was considered also the convergence of iterations. In accordance with the theorems of § 7.2 and § 9.4 when making these assumptions occurs the uniqueness of approximations/approaches in each stage, and iterations lead to certain minimum of the distance between sets  $X$  and  $Y$ . However, this minimum can prove to be local. So that the iterations would lead to the shortest distance (global minimum), the signal of zero approximation must be selected

sufficiently closely to  $x_{opt}$ .

The determination of zero approximation is the independent problem, which is reduced to the straight/direct resolution the formulated earlier variational problem, see relationships/ratios (9.17)-(9.19). We will consider the asymptotic method of its decision, suitable for the high contraction coefficients. This decision has much in common with the appropriate methods synthesis ChM and is of independent interest.

#### 9.7. Asymptotic synthesis FM of signals.

Above established/installed, that the synthesis of optimum FM signal  $x_{opt}$  is reduced to finding of optimum generating signal  $y_{opt}$  nearest to set  $X$ , and this, in turn, requires the determinations of optimum phase spectrum  $\alpha_{opt}(\omega)$ , with which the coefficient of proximity (9.17) it reaches maximum. On the other hand, signal  $y_{opt}$  is located at the shortest distance from FM of signal  $x_{opt}$ . In accordance with the theorem of § 7.2 its phase spectrum is phase spectrum of FM signal, i.e., the odd function of frequency, see (9.3) and (9.21).

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We can therefore be bounded to the examination of odd phase spectra,

after assuming

$$a(-\omega) = -a(\omega); a(0) = 0.$$

Furthermore, the amplitude spectrum  $a(\omega)$  makes sense to assign only by even function of frequency, since the amplitude spectrum of any FM signal is even and the coefficient of proximity in the form (7.15) does not depend on odd component in  $a(\omega)$ . Under these conditions the generating signal  $y(t)$  is real. As a result, proposing also that  $a(\omega)$  is finite in the interval  $(-\Omega, \Omega)$ , we obtain instead of (9.17) and (9.18)

$$C(X, y) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} |y(t)| dt = \max. \quad (9.23)$$

$$y(t) = \frac{1}{\pi} \int_0^{\Omega} a(\omega) \cos[\alpha(\omega) - \omega t] d\omega \quad (9.24)$$

Task consists of the determination of function  $\alpha(\omega)$ , that realizes maximum  $C(X, y)$ . For the approximation calculus of integral (9.24) it is possible to use the method of steady state. This is connected with one more assumption: one should consider that the derivative of the unknown phase spectrum  $\alpha'(\omega)$  varies monotonically in interval  $(0, \Omega)$ . The aforesaid means that we will seek optimum FM signal from the subset, subordinated to further condition. We must also explain, how this a limitation is dangerous from the point of view of the loss of the best signals, which ensure high degree of approximation.

Thus, counting for the concreteness, that the function  $\alpha'(\omega)$

monotonically grows <sup>1</sup>, we can use the formula of the method of steady state (8.24) exactly so, as this was done during the conclusion/output of relationships/ratios (8.31) and (8.42).

FOOTNOTE <sup>1</sup>. It is possible to take that decreasing  $\alpha'(\omega)$ , this leads only to the inversion of the obtained signal, the opposite reference direction of time. ENDFOOTNOTE.

As a result it is obtained

$$y(t) \approx \sqrt{\frac{2}{\pi}} \frac{a(\omega_0)}{\sqrt{a''(\omega_0)}} \cos[\alpha(\omega_0) - \omega_0 t + \pi/4], \quad (9.25)$$

moreover frequency  $\omega_0$  is connected with the current time  $t$  with stability condition of the phase:

$$\alpha'(\omega_0) = t; \quad 0 < \omega_0 < \Omega. \quad (9.26)$$

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Formula (9.23) leads further to the relationship/ratio

$$C(X, y) \approx \sqrt{\frac{2}{\pi T}} \int_{-T/2}^{T/2} \frac{a(\omega_0)}{\sqrt{a''(\omega_0)}} |\cos[\alpha(\omega) - \omega_0 t + \pi/4]| dt.$$

Here the sign of approximation/approach indicates the error, connected with the method of steady state. It is below, allowing/assuming also some other errors, we will not first write out the appropriate corrections. The evaluation/estimate of these approximations/approaches is done more lately.

Using dependence of  $t$  on  $\omega_c$ , by expressed formula (9.26), let us switch over in the latter/last integral to variable/alternating  $\omega_0$ . From (9.26) we have  $dt/d\omega_0 = \alpha''(\omega_0)$ ; therefore

$$C(X, y) \approx \sqrt{\frac{2}{\pi T}} \int_{\Omega_1}^{\Omega_2} a(\omega) \sqrt{\alpha''(\omega)} |\cos[\alpha(\omega) - \omega\alpha'(\omega) + \pi/4]| d\omega. \quad (9.27)$$

Is here omitted index in variable/alternating  $\omega$ , and integration limits  $\Omega_1$  and  $\Omega_2$  correspond to the boundaries of the signal:

$$\alpha'(\Omega_1) = -T/2, \alpha'(\Omega_2) = T/2. \quad (9.28)$$

Furthermore, taking into account (9.26), is assumed to be that performed condition  $0 \leq \Omega_1, \Omega_2 \leq \Omega$ .

As a rule, the assigned amplitude spectrum  $a(\omega)$  is the positive flat function, which slowly varies in interval  $(0, \Omega)$ . A similar character can be assumed, also, in  $\alpha''(\omega)$ . However, latter/last factor under integral (9.27) has oscillatory structure.

Using the expansion

$$\cos z = \frac{1}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1} \cos 2nz \quad (9.29)$$

we we can isolate from this factor "constant component"  $2/\pi$ .

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One should expect that this component makes a main contribution to

integral (9.27), while rapidly fluctuating additions give small correction. Therefore, being limited thus far to the first term of series/row (9.29), we obtain

$$C(X, y) \approx \frac{2}{\pi} \sqrt{\frac{2}{\pi T}} \int_{\Omega_1}^{\Omega_2} a(\omega) \sqrt{a''(\omega)} d\omega. \quad (9.30)$$

It is now not difficult to find the function  $a''(\omega)$ , with which the coefficient of proximity  $C(X, y)$  has a maximum. Applying Schwarz-Buniakowski's inequality, we obtain, taking into account (9.28) and standardization condition:

$$\begin{aligned} C^2(X, y) &\leq \frac{8}{\pi^2 T} \int_{\Omega_1}^{\Omega_2} a''(\omega) d\omega \int_{\Omega_1}^{\Omega_2} a^2(\omega) d\omega = \\ &= \frac{8}{\pi^2 T} [a'(\Omega_2) - a'(\Omega_1)] \int_{\Omega_1}^{\Omega_2} a^2(\omega) d\omega = \\ &= \frac{8}{\pi^2} \int_{\Omega_1}^{\Omega_2} a^2(\omega) d\omega \leq \frac{8}{\pi^2} \int_0^{\Omega} a^2(\omega) d\omega = \frac{8}{\pi^2}. \end{aligned} \quad (9.31)$$

Here there are two inequalities. The first of them is converted into the equality, if factors under integral (9.30) are proportional, i.e.

$$a''(\omega) = \gamma^2 a^2(\omega). \quad (9.32)$$

The second inequality becomes equality, only if integration limits comprise

$$\Omega_1 = 0, \Omega_2 = \Omega.$$

So that the coefficient of proximity would achieve maximum, it is necessary to take both these conditions. Factor of proportionality  $\gamma$  it is easy to determine further, if we integrate equation (9.32). This it gives

$$\int_0^{\pi} a''(\omega) d\omega = a'(\Omega) - a'(0) = T = \gamma^2 \int_0^{\pi} a^2(\omega) d\omega = \gamma^2 \pi.$$

Moreover here are again taken into consideration relationships/ratios (9.28). Consequently,  $\gamma = \sqrt{T/\pi}$ .

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Thus, optimum phase spectrum satisfies the equation

$$a''(\omega) = \frac{T}{\pi} a^2(\omega) \quad (9.33)$$

under the initial conditions  $a'(0) = -T/2$  and  $a(0) = 0$ .

FOOTNOTE 1. Latter/last condition follows from the odd parity of optimum phase spectrum  $a(\omega)$  and its continuity. ENDFOOTNOTE.

Direct substitution in (9.27) shows that value  $C$  actually/really reaches in this case a maximally possible value, determined by inequality (9.31):

$$C(X, Y) = \frac{2}{\pi} \sqrt{\frac{2}{\pi T}} \sqrt{\frac{T}{\pi}} \int_0^{\pi} a^2(\omega) d\omega = \frac{2\sqrt{2}}{\pi}. \quad (9.34)$$

Fig. 9.5 illustrates these results. In the upper part of the figure is shown the assigned spectrum of power  $a^2(\omega)$ .

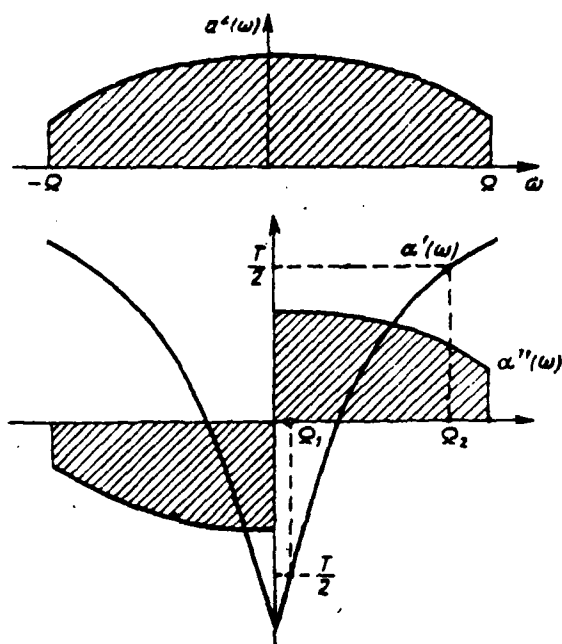


Fig. 9.5.

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In the lower part is given odd function  $\alpha''(\omega)$  whose each half is proportional to this spectrum in accordance with (9.33). Is there shown function  $\alpha'(\omega)$  and are noted integration limits  $Q_1$  and  $Q_2$ , which satisfy condition (9.28). The value of the coefficient of proximity is proportional to integral of  $a^2(\omega)$  within the limits indicated and reaches maximum with  $Q_1=0$  and  $Q_2=Q$ . Obviously, we arrived at the relationships/ratics, very close to the case of ChM signals. Equation (9.33) is similar (to 8.37) for rectangular

envelope. Furthermore ChM signals with symmetrical modulation, examined in § 8.6, they gave accurately the same value of the maximum coefficient of the proximity (see (8.47)). There is suggested the assumption that there are certain internal connection between ChM and FM oscillations. This connection/communication will be actually/really revealed subsequently.

Let us now point out only that a structure of ChM oscillation has the optimum generating signal, which satisfies the previous conditions. Actually/really, with  $|t| \leq T/2$  this signal is determined by relationship/ratio (9.25), which taking into account (9.33) and (9.26) acquires the form

$$y_{opt}(t) \approx \sqrt{\frac{2}{T}} \cos[\Phi(\omega) + \pi/4]. \quad (9.35)$$

Here

$$\begin{aligned} \Phi(\omega) &= \alpha(\omega) - \omega \alpha'(\omega) = \alpha(0) - \int_0^{\omega} \omega \alpha''(\omega) d\omega = \\ &= -\frac{T}{\pi} \int_0^{\omega} \omega \alpha^2(\omega) d\omega, \end{aligned} \quad (9.36)$$

but frequency  $\omega$  is connected with time  $t$  with relationship/ratio (9.26), which can be registered also in the form

$$\begin{aligned} t &= \alpha'(\omega) = \alpha'(0) + \int_0^{\omega} \alpha''(\omega) d\omega = \\ &= -\frac{T}{2} + \frac{T}{\pi} \int_0^{\omega} \alpha^2(\omega) d\omega. \end{aligned} \quad (9.37)$$

Oscillations of type (9.35), which have the structure

$$y(t) = A(t) \cos z(t).$$

we will further call real ChM signals (in contrast to usual ChM oscillations for which is characteristic the representation of the form  $A(t)e^{jk(t)}$  <sup>1</sup>).

FOOTNOTE <sup>1</sup>. However, then it is possible to name also FM signals of variable amplitude, since the phase takes values of 0 or  $\pi$  depending on the sign of cosine. Let us recall also that we everywhere deal concerning composite envelope, but not with strictly the signal.

ENDFOOTNOTE.

Instantaneous frequency of this exists  $z'(t)$ . For generating signal (9.35) instantaneous frequency vary monotonically. Actually/really, taking into account the previous relationships/ratios, we obtain

$$\begin{aligned} \omega_c(t) &= \frac{d\Phi}{dt} = \frac{d\Phi}{d\omega} \frac{d\omega}{dt} = \\ &= - \frac{T}{\pi} \omega a^2(\omega) / \frac{T}{\pi} a^2(\omega) = -\omega \end{aligned}$$

formula (9.37) it shows further that  $\omega_c(t)$  is a monotonic function.

Thus, which generates for the optimum FM signal is real ChM signal of constant amplitude with the monotone law of a change in the frequency.

In Fig. 9.6 is clarified the methodology of synthesis, which

directly escape/ensues from the obtained relationships/ratios. Optimum FM signal  $x_{opt}$  is nearest to  $y_{opt}$ . Its moments/torques of commutation correspond to zero  $y_{opt}(t)$ , i.e.

$$y_{opt}(t_k) = \sqrt{\frac{2}{T}} \cos \left[ \Phi(\omega_k) + \frac{\pi}{4} \right] = 0.$$

Therefore, after constructing of the assigned amplitude spectrum function  $\Phi(\omega)$ , in accordance with (9.36), we must determine values  $\omega_k$ , with which it is implemented latter/last condition. These values determine, in turn, moments/torques  $t_k$  according to equation (9.37), and also the unknown FM signal, shown in the right side of the figure. Characteristically monotone condensation of the moments/torques of commutation toward the end of the signal, connected with the assumption about the monotonicity  $\alpha'(\omega)$ .

Let us explain now, what degree of approximation gives this method of synthesis. The distance between signals  $x_{opt}$  and  $y_{opt}$  is determined by the obtained coefficient of proximity (9.34):

$$d_{min}^2 = 2[1 - C(X, Y)] = 2(1 - 0.9) = 0.2. \quad (9.38)$$

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The method examined has as a goal to approach autocorrelation functions, and it is important to consider, how this is reached. In § 7.2 was obtained relationship/ratio (7.25), which establishes the approximate dependence between a root-mean-square error in the autocorrelation functions and with a distance of  $d_{min}$ .

This relationship/ratio takes the form

$$\delta_{\min} \approx \sqrt{2m} d_{\min}.$$

Therefore, taking into account (9.38), we obtain

$$\delta_{\min} \approx \sqrt{0.4m} \approx 0.63 \sqrt{m}. \quad (9.39)$$

This result has fundamental value. We can claim that during the large compression the remainders/residues of autocorrelation function in the optimum case are of the order  $1/\sqrt{m}$ . Until now, this conclusion followed only from the analysis of known signals <sup>1</sup>.

FOOTNOTE <sup>1</sup>. During the conclusion/output of relationship/ratio (7.25) were assumed some average/mean conditions. For the best signals the root-mean-square level of remainders/residues can be less (9.39), but it has the same order relative to value  $m$ .

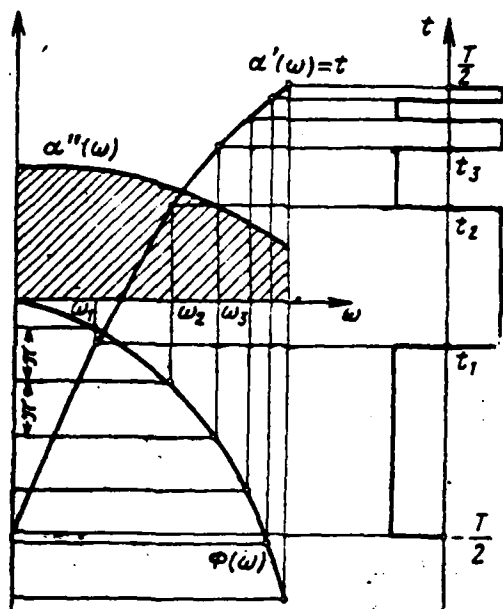


Fig. 9.6.

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## 9.8. Another treatment of method.

As it was mentioned, the generality of the methods of synthesis makes it possible to assure certain connection/communication between ChM and FM oscillations. This connection/communication lies in the fact that any FM signal it is possible, it proves to be, to represent in the form of the imposition of the corresponding ChM signals.

In order to show this, let us begin from formula (9.3) for the spectrum FM of the oscillation

$$\tilde{x}(\omega) = \frac{2}{j\omega VT} \left\{ \frac{e^{-j\omega t_0}}{2} + \sum_{k=1}^{N-1} (-1)^k e^{-j\omega t_k} + \right. \\ \left. + (-1)^N \frac{e^{-j\omega t_N}}{2} \right\}. \quad (9.40)$$

Let us introduce instead of the index of summation  $k$  continuous the variable/alternating  $z$  and we will consider that there is a continuous function  $t(z)$ , which takes values  $t_k$  at points  $z=k=0, 1, \dots, N$ . Then it is possible to use the summation formula of Poisson (see for example [1])

$$\sum_{k=-\infty}^{\infty} \frac{\varphi(k+0) + \varphi(k-0)}{2} = \sum_{v=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(z) e^{j2\pi v z} dz \quad (9.41)$$

for converting expression (9.40). Actually/really, if we determine function  $\varphi(z)$  by the relationship/ratio

$$\varphi(z) = \begin{cases} \exp \{-j[\omega t(z) - \pi z]\} & \text{при } 0 \leq z \leq N \\ 0 & \text{при } z < 0 \text{ и } z > N. \end{cases}$$

Key: (1). with. (2). and.

that, as can easily be seen, left side (9.41) is converted into the expression, included in the brackets in formula (9.40), and we obtain on the basis of Poisson's formula:

$$\tilde{x}(\omega) = \frac{2}{j\omega VT} \sum_{v=-\infty}^{\infty} \int_0^N \exp \{-j[\omega t(z) - (2v+1)\pi z]\} dz.$$

Since  $t(z)$  - monotonic function, it is possible to pass in the latter/last integral to by the variable/alternating  $t$ . As a result

after the series/row of the conversions (is used integration in parts, the terms outside the integral vanish with the summation) is obtained

$$\tilde{x}(\omega) = \frac{2}{\pi \sqrt{T}} \sum_{v=-\infty}^{\infty} \frac{1}{2v+1} \int_{-T/2}^{T/2} \exp \{j[(2v+1)\pi z(t) - \omega t]\} dt. \quad (9.42)$$

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Finally, banking in pairs the members of this sum, it is easy to obtain

$$\tilde{x}(\omega) = \frac{4}{\pi \sqrt{T}} \sum_{v=0}^{\infty} \frac{1}{2v+1} \int_{-T/2}^{T/2} \sin \{[(2v+1)\pi z(t)]\} e^{-j\omega t} dt. \quad (9.43)$$

It is now clear that in the interval  $(-T/2, T/2)$  of PM signal  $x(t)$  can be presented in the form of the infinite sum of real ChM signals:

$$x(t) = \frac{4}{\pi \sqrt{T}} \sum_{v=0}^{\infty} \frac{1}{2v+1} \sin \{[(2v+1)\pi z(t)]\}. \quad (9.44)$$

This result it is not difficult to interpret. Let us consider the segment of rectangular oscillation (meander), shown in Fig. 9.7a. As argument here serves value  $z$  and jumps occur at the whole values of  $z=0, 1, 2, \dots, N$ . In the interval  $0 < z < N$  this oscillation can be decomposed in the usual Fourier series on the sines. The fundamental harmonic of this resolution is also depicted in figure. It is not difficult to ascertain that the coefficients of this series/row the same as in formula (9.44). Since the variable/alternating  $z$  nonlinearly depends on time, then on scale  $t$  jumps occur through the

unequal gaps/intervals and is obtained FM signal, shown in Fig. 9.7b. On the other hand, the nonlinear dependence of  $z$  on  $t$  leads to the fact that each harmonic of Fourier series is converted in ChM oscillation, but this complication, obviously, in any way does not affect the coefficients of series/rcw.

Let us examine in more detail the first "harmonic" (maximum in the amplitude)

$$x_1(t) = \frac{4}{\pi \sqrt{T}} \sin \pi z(t).$$

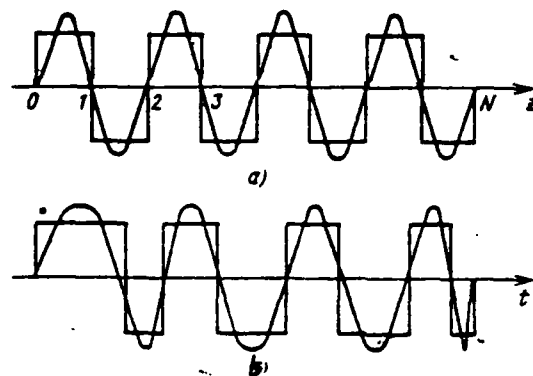


Fig. 9.7.

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Of the condition, function  $z(t)$  takes the whole values of  $z=k=0, 1, 2, \dots, N$  at the moments of the commutation of phase  $\psi$ . ChM oscillation/vibration  $x_1(t)$  passes at these moments/torques through zero. Consequently, we obtained nothing else but another form of the recording of generating signal (9.35). In accordance with the rule of standardization it is necessary to only change the amplitude of this signal so that its energy would be equal to unity. If we disregard/neglect integral of rapidly-vibrating component, this standardization gives

$$y(t) = \sqrt{\frac{2}{T}} \sin \pi z(t). \quad (9.45)$$

This completely will be coordinated with (9.35).

In light of this new representation method examined above of synthesis obtains the following treatment. Intending to find the FM signal which approaches the assigned amplitude spectrum  $a(\omega)$ , we select its the first "harmonic" (9.45) and we assume that in certain approximation/approach the spectrum of entire signal corresponds to the spectrum of this "harmonic". It is further necessary to determine

phases  $z(t)$  so as to fulfill the assigned amplitude spectrum. Logically, we come to the problem of synthesis of ChM oscillation/vibration and we use a method of steady state. In accordance with the results of Chapter 8 is selected the law of a change in the instantaneous frequency so that the deviation coincides with the width of the assigned spectrum, and a change in the rate of modulation provides the necessary structure of the latter. This explains the similarity of the methods of synthesis of ChM and FM. Certain difference is connected with the fact that signal (9.45) real and consists of two ChM usual type oscillations/vibrations:

$$y(t) = \frac{1}{j\sqrt{2T}} e^{j\pi z(t)} - \frac{1}{j\sqrt{2T}} e^{-j\pi z(t)}.$$

If instantaneous frequency  $\omega_c(t) = \pi z'(t)$  vary within the range of 0 to  $\Omega$ , then in the approximation/approach of the method of steady state first component/term/addend describes the form of the spectrum in the band from 0 to  $\Omega$ , and the second - from 0 to  $-\Omega$ . This will be in complete agreement with that presented earlier (see Fig. 9.5).

This treatment makes it possible to come to light/detect/expose the series/row of important positions. In particular, we can consider the minimum interval between the commutations. If the assigned spectrum has higher frequency  $\Omega$ , then the greatest instantaneous frequency of ChM signal (9.45) is also equal to  $\Omega$  (in accordance with the principle of steady state), and the smallest half-period of

oscillation/vibration comprises  $\pi/\Omega$ . The moments/torques of commutation  $M_k$  are determined by zero "sinusoids" (9.45); therefore  $\Delta t_{min} \approx \pi/\Omega$  1.

FOOTNOTE 1. The given evaluation/estimate is of certain interest for the iterative process, described in §9.6. It is not difficult to comprehend that a number of commutations  $N$  can be changed from one space to the next and appears the fear that number  $N$  will with the iterations unlimited grow. Then to be obtained the unrealizable virtually signal. On the basis of that presented it is possible, however, to claim that this it will not happen, since the minimum interval between the commutations is limited by the higher frequency of the spectrum. ENDFOOTNOTE.

The average value of the interval between the commutations approximately/exemplarily corresponds to average rapid of the spectrum  $\Omega/2$ , i.e.,  $\Delta t_{av} \approx 2\pi/\Omega$ .

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Consequently, total number of commutations on duration  $T$  obtains evaluation/estimate  $N \approx T/\Delta t_{av} \approx \Omega T/2\pi = m/2$ . This will be coordinated with the known property of "good" FM signals: a number of alternations in the sign is approximately half of samples.

We can in a new way throw light also assumption about the monotonicity of derived  $\alpha'(\omega)$ . As it was noted, this assumption is equivalent so that the instantaneous signal frequency (9.45) monotonically depends on time. This limitation is substantial only during the determination of law ChM, which ensures the assigned amplitude spectrum. Simply we do not know another locked method of synthesis of ChM signals, besides the method of steady state, but the latter gives the foreseeable solution only with this restriction. Resolution of FM signal into the "harmonics" (9.44) is useful during any arrangement/position of commutations, and, if we could construct nonmonotonic ChM signal with the assigned spectrum, then would be obtained the corresponding nonmonotonic FM signal.

The main error in the approximation/approach, which leads to the final distance between sets X and Y, is connected with the replacement of "sinusoidal" oscillation (9.45) by square wave (FM signal); in other words, with the neglect of all terms of series/row (9.44), except the first. But this approximation/approach does not depend on monotonicity or nonmonotonicity of the law of modulation. Therefore one should expect that nonmonotonic FM signals must not give substantially best approximation/approach to the assigned spectrum, than monotone ones, examined higher.

Let us show this more strictly. Any real signal can be presented in the form

$$y(t) = A(t) \cos \phi(t). \quad (9.46)$$

Assuming this signal to be rapidly vibrating, it is isolated the slowly varying enveloping  $A(t) > 0$  and carry rapidly-vibrating structure to the second factor. Then the coefficient of proximity (9.23) obtains the form

$$C(X, y) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} A(t) |\cos z(t)| dt.$$

If we again use expansion (9.29), then without taking into account integrals of rapidly-vibrating components/terms/addends the coefficient of proximity will depend only on amplitude envelope

$$C(X, y) = \frac{2}{\pi \sqrt{T}} \int_{-T/2}^{T/2} A(t) dt.$$

In order to obtain maximum, it is necessary to fit optimum  $A(t)$ . For this purpose let us again use Schwarz inequality - Bunyakowski:

$$\begin{aligned} C^2(X, y) &\leq \frac{4}{\pi^2 T} \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} A^2(t) dt = \\ &= \frac{4}{\pi^2} \int_{-T/2}^{T/2} A^2(t) dt. \end{aligned}$$

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First/last integral can be expressed through the energy of

signal  $y(t)$ . If we disregard/neglect integral of the rapidly-vibrating function, then it is obtained

$$\int_{-T/2}^{T/2} A^2(t) dt = 2 \int_{-T/2}^{T/2} |y(t)|^2 dt = 2.$$

Thus,  $C^2(X, y) \leq 8/\pi^2$  and, as usual, upper bound reaches at the proportionality of factors, i.e., under the condition

$$A(t) = \text{const.}$$

This means that the real generating signal with rectangular envelope makes it possible to obtain better approximation/approach by FM signal in comparison with all other signals of type (9.46).

Any amplitude changes make the quality worse of approximation. However, if envelope is rectangular, then, independent of the character of phase modulation, attains the limiting value of the coefficient of proximity  $C(X, y) = 2\sqrt{2}/\pi$ . The aforesaid means that nonmonotonic FM signals cannot give the best approximation/approach to the assigned spectrum, than monotonic ones examined above<sup>1</sup>.

FOOTNOTE <sup>1</sup>. This conclusion/output is valid with that degree of accuracy that is accepted above. If we take into account corrections to the asymptotic solution, monotone signals can prove to be non-optimal. ENDFOOTNOTE.

It is nevertheless interesting to explain, what structure have "good" FM signals, obtained by other methods; are encountered among them monotone. Analysis shows that the monotonicity actually/really occurs for many such signals.

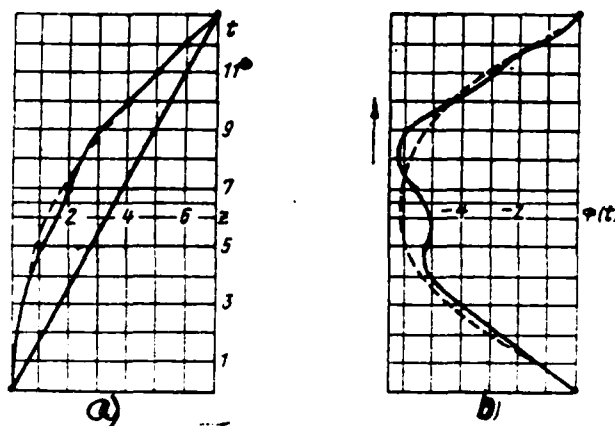


Fig. 9.8.

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In Fig. 9.8a according to discrete/digital values  $t_z$  is constructed the plotted function  $t(z)$  for Barker's 13-digit signal. Is there depicted the smoothed curve, in which is absent certain superimposed fluctuation. This curve is convex, it does not change the sign of curvature. More clearly this structure is visible in Fig. 9.8b, where are shown the same curves minus linear component, i.e., is constructed function  $\Phi(t) = \pi z(t) - \pi N(t/T + 1/2)$ , and its "averaging".

It is obvious, in certain approximation/approach  $\Phi(t)$  there is the symmetrical convex function, close to the quadratic parabola. This means that each term of series/row (9.42) corresponds to the spectrum of ChM signal with the monotone (in the case of parabola -

linear) law of a change in the frequency. In Fig. 9.9 analogously constructed function  $\phi(t)$  for a 43-digit FM signal, based on Legendre's symbols. It is obvious, the general/common/total structure of curve here the same as in the preceding case.

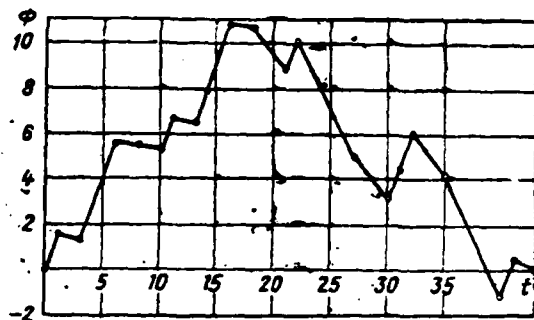


Fig. 9.9.

## 9.9. Estimation of error in the asymptotic solution.

From previous is clear the nature of the errors for the method of synthesis examined. These errors are connected, in the first place, with an error in the method of steady state, used for the determination of the generating ChM signal, and, in the second place, with the neglect of all terms of series/rcw (9.29), except the first. The first reason to equal measure relates to the synthesis of ChM signals, it is exhibited, in particular, in the Fresnel pulsations, examined in §8.5. The second reason is specific for FM. As we saw, she was equivalent to the replacement of the stepped structure of that shown in Fig. 9.7, smoothly curve.

It is possible to consider an error in the approximation/approach, if we determine more accurately the value of

the coefficient of proximity, which is in our method the measure of the quality of approximation. Here it is possible to discuss as follows.

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The examined method of synthesis led to the generating signal

$$y(t) = \sqrt{\frac{2}{T}} \cos[\Phi(t) + \pi/4]; \quad -T/2 < t < T/2. \quad (9.47)$$

where the function  $\phi(t)$  was determined in the parametric form of relationship/ratio (9.36)-(9.37). In accordance with precise formula (9.13) the coefficient of proximity for this signal and set of FM signals  $X$  comprises

$$C(X, y) = \frac{\sqrt{2}}{T} \int_{-T/2}^{T/2} \cos \left[ \Phi(t) + \frac{\pi}{4} \right] dt.$$

If we again use expansion (9.29), then the first term immediately will lead to obtained previously value  $C = 2\sqrt{2}/\pi$ , and the others will give the corresponding correction. As a result

$$C(X, y) = \frac{2\sqrt{2}}{\pi} + \sum_{\mu=1}^{\infty} C_{\mu}.$$

where

$$C_{\mu} = \frac{4\sqrt{2}}{\pi T} \frac{(-1)^{\mu-1}}{4\mu^2 - 1} \int_{-T/2}^{T/2} \cos 2\mu \left[ \Phi(t) + \frac{\pi}{4} \right] dt.$$

Using dependence (9.37), it is possible to switch over in this

integral to variable/alternating  $\omega$ . This it gives:

$$C_\mu = \frac{4\sqrt{2}}{\pi^2} \frac{(-1)^{\mu-1}}{4\mu^2-1} \int_0^\pi a^2(\omega) \cos 2\mu \left[ \Phi(\omega) + \frac{\pi}{4} \right] d\omega. \quad (9.48)$$

Moreover function  $\phi(\omega)$  is defined by specific relationship (9.36).

For calculating the integral let us use the method of steady state. Stationary point  $\omega_0$  is determined by condition  $\Phi'(\omega_0) = -\omega_0 a^2(\omega_0) = 0$ .

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Consequently,  $\omega_0 = 0$ , and the method of steady state leads to the expression (are here omitted some intermediate conversions)

$$C_\mu = \frac{2\sqrt{2}}{\pi} \frac{a(0)}{\sqrt{\pi} (4\mu^2-1)\sqrt{2\mu}} \left\{ \cos(2\mu-1) \frac{\pi}{4} + O(1/\sqrt{m}) \right\} \approx \frac{2\sqrt{2}}{\pi\sqrt{m}(4\mu^2-1)\sqrt{2\mu}} \{1 + O(1/\sqrt{m})\}.$$

where it is also considered that  $\frac{a(0)}{\sqrt{\pi}} \approx \frac{1}{\sqrt{\pi}} \sqrt{\frac{\pi}{2}} = \frac{1}{\sqrt{2}}$ .

Further we have

$$\sum_{\mu=1}^{\infty} \frac{1}{(4\mu^2-1)\sqrt{2\mu}} = \frac{1}{\sqrt{2}} \left( \frac{1}{3} + \frac{1}{15\sqrt{2}} + \frac{1}{35\sqrt{3}} + \dots \right) \approx 0.28$$

and it is final

$$C(X, y) = \frac{2\sqrt{2}}{\pi} \left\{ 1 + \frac{0.28}{\sqrt{m}} + O(1/m) \right\}. \quad (9.49)$$

Thus, an error in the asymptotic solution is of the order  $1/\sqrt{m}$  and the less, the greater the contraction coefficient. Let us emphasize, however, that here has in mind the quadratic approximation/approach to the assigned amplitude spectrum, evaluated by the value of the coefficient of proximity. With this criterion of synthesis the asymptotic solution can be, apparently, considered satisfactory, beginning with  $m \sim 50$ , when relative error does not exceed 5%. For the minimax criterion, and especially for the quantified FM signals, this solution requires further refinements.

9.10. The refinement of asymptotic approximation/approach by successive design.

One of the methods of refining the asymptotic approximation/approach are iterations by method of the successive design, which in connection with the task of synthesis of FM signals being investigated are in detail examined in §9.6. Are given below some results, obtained by this method.

The desired spectrum of power  $a^2(\omega)$ , to which was produced the approximation/approach, was assigned in the form of the Hemming function

$$a^2(\omega) = 1 + \epsilon \cos \omega; \quad -\pi < \omega < \pi. \quad (9.50)$$

As it was noted in §8.5, this form of the spectrum gives

satisfactory approximation/approach to an optimum and it does not lead to the excessive complication. Upper bound of the spectrum is here accepted equal to  $\pi$ . This does not break generality, if the duration of signal is numerically equal to contraction coefficient ( $T=m$ ). Parameter  $g$  it is expedient to select in such a way that on its average part spectrum (9.50) was close to the spectrum of the ideal compressed impulse/momentum/pulse - rectangular the sample of single duration. This leads to values of  $g \sim 0.25-0.5$ .

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For the spectrum of the form in question the general formulas of asymptotic method (9.35)-(9.37) give

$$y(t) = \sqrt{\frac{2}{m}} \cos \left\{ \frac{m}{\pi} \left[ \frac{\omega^2}{2} + g(\omega \sin \omega + \cos \omega - 1) \right] - \frac{\pi}{4} \right\},$$

$$0 < \omega < \pi.$$

moreover  $\omega$  is connected with time  $t$  with the dependence

$$t = \frac{m}{\pi} (\omega + g \sin \omega).$$

In the latter/last relationship/ratio is not taken into consideration the unessential shift/shear of entire signal in time to value  $T/2$ , cm (9.37). According to these formulas were calculated moments/torques  $\mu_k$  in which the generating signal  $y(t)$  it is converted into zero. Values  $\mu_k$  determine the signal of zero approximation. These values correspond to the roots of the equation

$$\frac{\omega^2}{2} + g(\omega \sin \omega + \cos \omega - 1) = \frac{\pi}{m} \left( k\pi - \frac{\pi}{4} \right); \quad k = 1, 2, \dots$$

which was solved by the method of polecat during the division of interval  $0 \leq \omega \leq \pi$  into 512 parts. The next step reduced to the determination of amplitude spectrum  $|x(\omega)|$  of the found FM signal and coefficient of proximity  $C(x, Y)$ , which characterizes degree of approximation to the assigned spectrum. Amplitude spectrum  $|x(\omega)|$  was calculated from formulas (9.20). In accordance with the theorem of §7.2, see formula (7.15), the coefficient of proximity has a value

$$C(x, Y) = \frac{1}{\pi} \int_0^{\pi} a(\omega) |\tilde{x}(\omega)| d\omega.$$

This integral was computed from Simpson's rule, also with the division of interval into 512 parts. Preliminary check showed the accuracy of this calculation on the order of 4-5 signs after comma.

After the calculation of the signal of zero approximation and its characteristics was implemented the first space of iterations. The generating signal of first approximation  $y_1(t)$  was determined by integral of Fourier (9.21). Calculation was conducted through Simpson's rule into 8 m the points of interval  $(0, \pi)$ . Further by the method of polecat were determined values  $t_k^{(1)}$ , in which  $y_1(t)$  it is converted into zero. These values are moments/torques of commutation of FM signal of first approximation. As earlier, was calculated the spectrum, the coefficient of proximity and the correlation function

of the obtained signal. Then all calculations were repeated in order to determine the signal of the second approximation/approach and its characteristic, etc.

Table 9.1 depicts the results of these calculations. Are here for some  $m$  given the values of the coefficient of proximity  $C$ , obtained with the consecutive iterations. Furthermore, are indicated the maximum remainders/residues of the normalized autocorrelation function  $\mu$ , and also the value of coefficient of  $k$  which was used in §9.2 for the evaluation/estimate of the level of remainders/residues (with respect to  $\sqrt{m}$ ).

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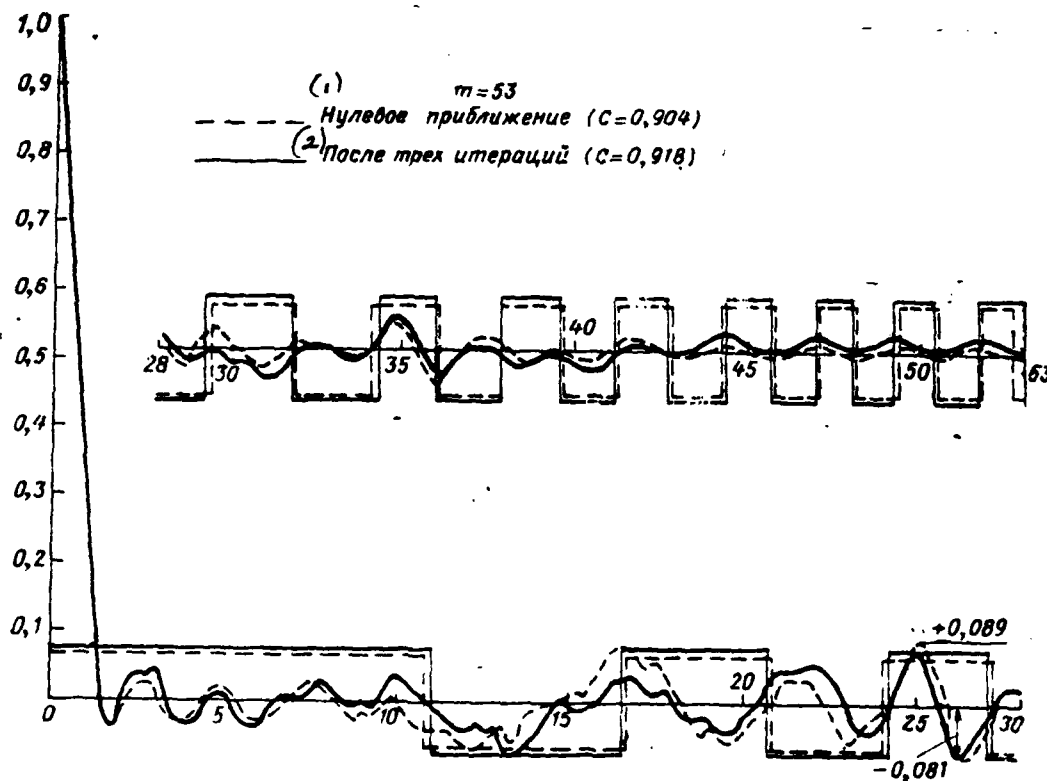


Fig. 9.10a.

Key: (1). Zero approximation. (2). After three iterations.

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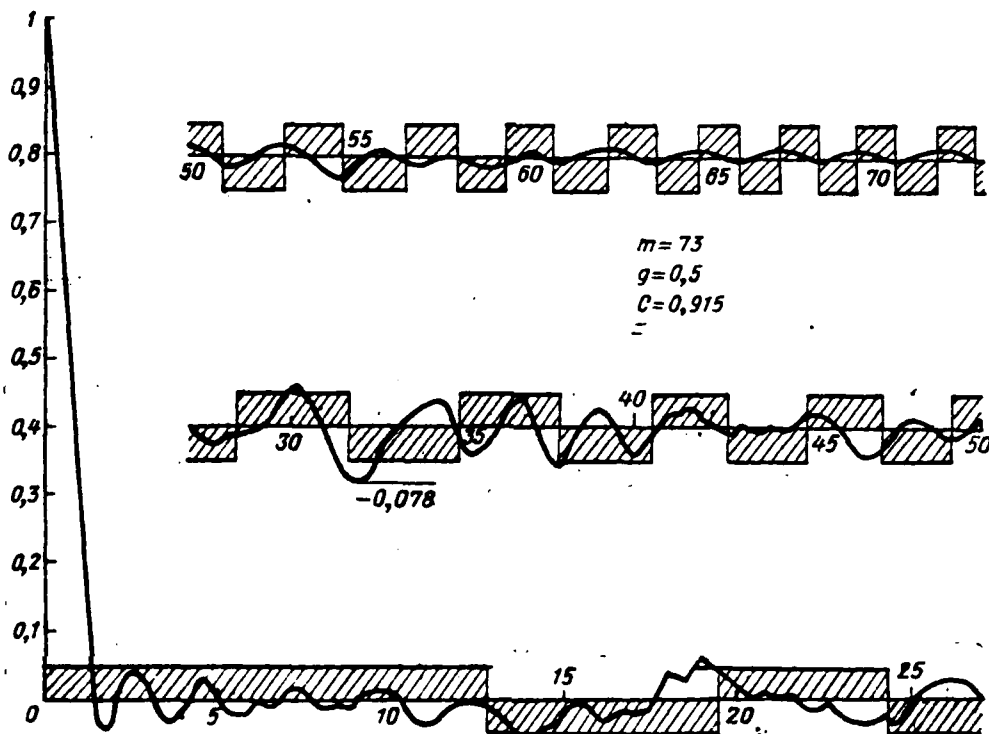


Fig. 9.10b.

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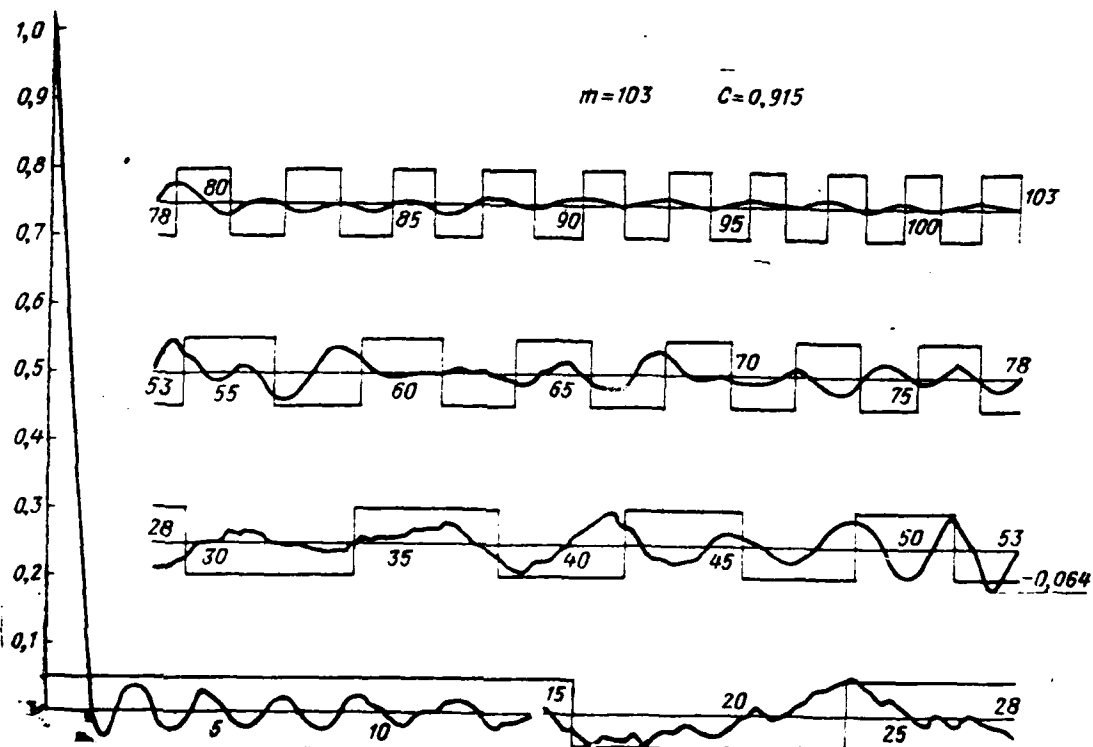


Fig. 9.10c.

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The analysis of the data indicated confirms fundamental theoretical results. The coefficient of proximity monotonically increases with the iterations. The theoretical evaluation/estimate of this value  $C=2\sqrt{2}\pi=0.900$  is completely satisfactory.

Maximum remainder/residue is changed with the iterations

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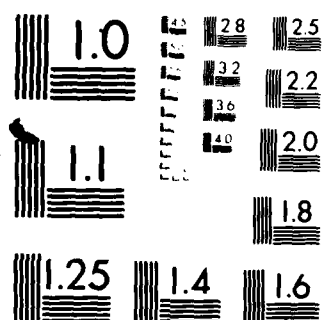
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irregularly, it can both be reduced and increase. Of course this occurs because the method of synthesis uses quadratic approximations/approaches, and the level of the greatest remainder/residue is only indirectly connected with the utilized criterion. But as a whole the level of maximum remainder/residue comprises  $(0.6+0.7)\sqrt{m}$ , which corresponds to the best known signals, given in §9.2. However, one should emphasize that we here synthesize the not quantized FM signals and comparison with those quantified, examined in §9.2, it is not completely justifiable/legitimate. The corresponding methods of the synthesis of KFM signals are given in Chapter 10.

Fig. 9.10 shows the obtained not quantized FM signals, and also their autocorrelation functions.

Table 9.2.

		Нулевое прибли- жение	Итерации (2)				
			1-я	2-я	3-я	4-я	5-я
$m = 20$	C	0,897	0,906	0,908	0,909	0,910	0,911
	$\mu$	0,124	0,133	0,129	0,126	0,114	0,116
	k	0,55	0,59	0,58	0,56	0,51	0,52
$m = 41$	C	0,903	0,909	0,910	0,911		
	$\mu$	0,101	0,093	0,098	0,089		
	k	0,65	0,59	0,63	0,57		
$m = 53$	C	0,904	0,910	0,914	0,918		
	$\mu$	0,089	0,088	0,087	0,081		
	k	0,65	0,64	0,63	0,59		
$m = 73$	C	0,908	0,915				
	$\mu$	0,079	0,079				
	k	0,67	0,68				
$m = 103$	C	0,909	0,915				
	$\mu$	0,071	0,064				
	k	0,79	0,65				

Key: (1). Zero approximation. (2). Iteration.

#### 9.11. Other iterative methods.

The method of successive design (projective-gradient) this is it goes without saying not the only iterative method, suitable for refining the obtained approximations/approaches. Furthermore, in the version examined this method is used only for the synthesis on the criterion of the proximity when is minimized the distance between X and Y and is provided approximation/approach to the assigned amplitude spectrum in sense (7.17). Other iterative methods make it possible to solve more general problems of synthesis of FM signals, to in particular find approximations/approaches to the unrealizable correlation functions.

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For example, it is possible to minimize the functional

$$f(x) = \int_{-\infty}^{\infty} [|F(t)| - |R(t)|]^2 dt, \quad (9.51)$$

where  $F(t)$  the arbitrary assigned (on the modulus/module) function, and  $R(t)$  - the autocorrelation function of the unknown FM signal. Are possible also other versions, when functional  $f(x)$  is connected in another adequate/approaching manner with the unknown signal. The characteristic feature of problem is the fact that the minimization must be produced on the set of FM signals, characterized for each moment of time only by sign ( $\pm 1$  or  $\pm 1/\sqrt{T}$ ), i.e. the permissible signals are rigidly limited<sup>1</sup>.

FOOTNOTE 1. Analogous tasks are encountered in the theory of optimum control [25]. To the not quantized FM signals correspond in this case the so-called relay steering functions. ENDFOOTNOTE.

When selecting of the method of synthesis should be considered the limitation indicated.

However, Krupitskiy and Sergeyenko [34] recently showed that it

is possible to bridge this difficulty and to construct iterations on the base of the usual gradient method, which does not assume the limitation of the permissible set. Their method is based on the fact that in accordance with (9.2) any FM signal is unambiguously assigned by its moments/torques of commutation  $t_k$  (see Fig. 9.1). Considering values  $t_k$  as the independent arguments, it is easy to note that the minimized functional  $f(x)$  is a function of a finite number of variable/alternating

$$f(x) = f(t_1, t_2, \dots, t_{N-1}) \quad (9.52)$$

the latter can take any values in the interval  $(-T/2, T/2)$ . Gradient method is used further for the minimization of this function of many variable/alternating.

On the base of this method were obtained the solutions of several problems of synthesis of FM signals for comparatively small compression ( $m < 25$ ). Among other things were conducted the iterations from the asymptotic initial approximations/approaches, examined above [61]. These calculations show, in particular, that asymptotic solution gives a comparatively good approximation/approach, it can be only a little improved.

But one should indicate certain nonoptimality of method, which escaped, apparently, from authors' attention. The recording of the functional being investigated in the form (9.52) leads in a number of

cases to the appearance of the local extrema which no during other representation of the unknown signal. Let us assume, for example, is required to find FM signal  $x(t)$ , which ensures best approximation to the assigned real signal  $y(t)$ . In this case the functional being investigated is a coefficient of the proximity

$$f(x) = C(x, y) = \int_{-T/2}^{T/2} y(t) x(t) dt \quad (9.53)$$

and it must be maximized on all  $x(t) \in X$ . We know (see §9.4) that unique solution of this task gives FM signal of the form

$$x(t) = \frac{1}{\sqrt{T}} \operatorname{sign} y(t).$$

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It is possible to show that if FM signals are represented not through the moments/torques of commutation, but in the form (9.1), at this solution it is possible to arrive, for example, by a projective-gradient method, functional (9.53) having sole maximum, and it is possible to begin from any initial signal. On the other hand, introducing the designation

$$\psi(t) = \int_0^t y(t) dt$$

and using (9.2), we come to the expression

$$f(x) = \frac{1}{2} \psi\left(-\frac{T}{2}\right) - \psi(t_1) + \psi(t_2) + \dots + \\ + (-1)^{N-1} \psi(t_{N-1}) + \frac{1}{2} (-1)^N \psi\left(\frac{T}{2}\right),$$

which corresponds to form (9.52). In the particular case of  $y(t) = \cos t$

(see Fig. 9.11a) gradient  $f'(x)$  is a vector with the components

$$f'_k = (-1)^{k-1} \frac{1}{\sqrt{T}} \cos t_k; \quad k=1, 2, \dots, N-1.$$

The maximization of functional (9.53) according to the method of operation [34] is reduced to changes  $x_k$  in the direction of gradient, i.e.,  $x_k$  increases, if  $f'_k > 0$  and vice versa. It is not difficult to see that the result of this maximization depends on initial signal. At the initial signal, shown in Fig. 9.11b, reaches the global maximum, which corresponds to Fig. 9.11c, but with the initial signal of the form Fig. 9.11d, method leads to the local maximum Fig. 9.11e.

Further the iterative methods in question use the representation of FM signals in the form (9.1) and, apparently, they do not have this deficiency/lack.

We will minimize functional (9.51) by a projective-gradient method, examined in §1.10. In this case the signal of next approximation/approach  $x^{(k+1)}(t)$  is formed from the signal of previous approximation/approach  $x^{(k)}(t)$  according to the rule

$$x^{(k+1)} = P_X[x^{(k)} - \alpha_k f'(x^{(k)})]. \quad (9.54)$$

Here  $P_X$  - operator design to the set of FM signals  $X$ . In accordance with the theorem of §9.4 this design (approximation) is reduced to the ideal limitation (see Fig. 9.2), so that

$$P_X[y(t)] = \frac{1}{\sqrt{T}} \operatorname{sign}\{\operatorname{Re} y(t)\}. \quad (9.55)$$

It is possible to show that the gradient of functional (9.51) has a value

$$J'(x) = 2 \int_{-T/2}^{T/2} [|R_x(t')| - |F(t')|] \operatorname{sign} R_x(t') x(t - t') dt', \quad (9.56)$$

where  $R_x(t)$  - correlation function of signal  $x(t)$ .

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Thus, construction of the minimizing sequence consists of the following operations:

1. Is taken initial FM signal  $x^{(0)}(t)$ .
2. Is computed direction of gradient according to (9.56).
3. Is chosen length of space  $\alpha$  and is implemented space on antigradient
 
$$p^{(1)} = x^{(0)} - \alpha J'(x^{(0)})$$
4. Signal  $p^{(1)}$  undergoes ideal limitation according to (9.55) for obtaining signal  $x^{(1)}$ .

Further process is repeated, beginning with p. 1. This process realizes approximation/approach to an arbitrary (unrealizable) autocorrelation function.

Analogous results can be obtained, applying the conditionally-gradient method, proposed by Dem'yanov [25]. During the use/application of this method for the synthesis of FM signals the minimizing sequence is constructed as follows.

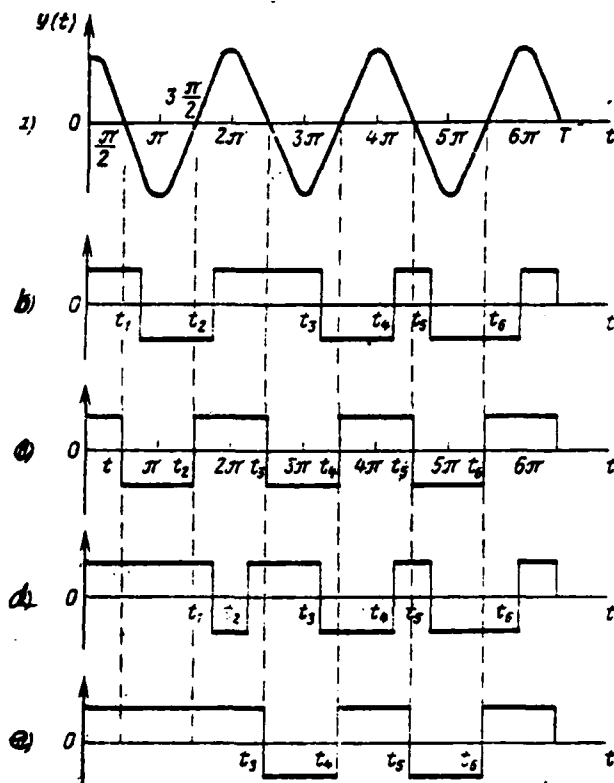


Fig. 9.11.

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If signal  $x^{(k)}(t)$  and gradient  $f[x^{(k)}(t)]$  do not coincide in the sign, then

$$x^{(k+1)}(t) = \overline{x^{(k)}(t)}.$$

If for certain interval of time  $p_i < t < q_i$ , signs  $x^{(k)}(t)$  and  $f[x^{(k)}(t)]$  are identical, the signal of the following approximation/approach is constructed according to the rule

$$\text{sign } x^{(k+1)}(t) = \begin{cases} \text{sign } x^{(k)}(t) & \text{if } p_i \leq t \leq p_i + \alpha(q_i - p_i); \\ -\text{sign } f[x^{(k)}(t)] & \text{if } p_i + \alpha(q_i - p_i) < t < q_i. \end{cases} \quad (9.57)$$

Key: (1). with.

where  $\alpha$  - space of iterations ( $0 < \alpha < 1$ ).

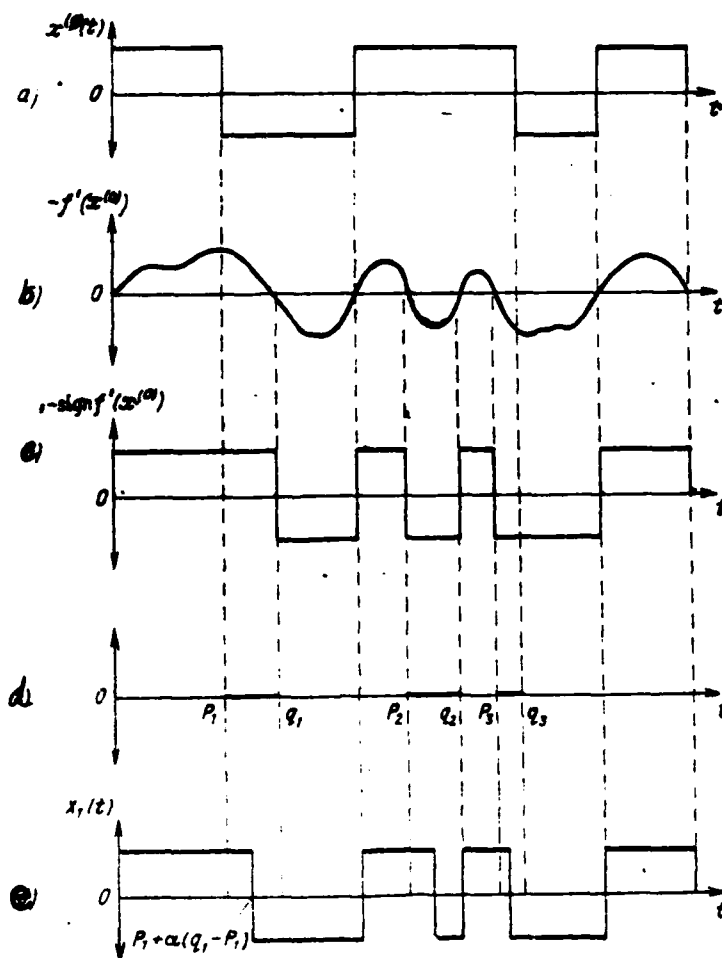


Fig. 9.12.

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Thus, construction of the minimizing sequence consists of the following (Fig. 9.12).

1. Is taken initial FM signal  $\hat{x}^{(0)}(t)$ .
2. Is computed gradient  $f'(x^{(0)})$  according to (9.56).
3. Obtained gradient undergoes ideal limitation (Fig. 9.12c).
4. Are noted intervals of time, for which signs  $\text{sign } f'(x^{(0)})$  and  $\text{sign } \hat{x}^{(0)}$  do not coincide (Fig. 9.12d).
5. On part of these segments, corresponding to (9.57), sign of signal vary by reverse/inverse (Fig. 9.12e is carried out for  $\alpha=1/2$ ). Further process is repeated, beginning with p. 1.

FOOTNOTE 1. A conditionally-gradient method let us use in general, on we presented its version, suitable for FM signals (two-position controls). ENDFOOTNOTE.

Both methods (design-gradient and conditionally-gradient) can be used, in the principle, for the minimization of any functional, not only form (9.51). Is changed in this case only formula for gradient (9.56). The major advantage of these methods over synthesis on the criterion of proximity consists, as already mentioned, in the fact that it is possible to find approximations/approaches to the unrealizable properties, in particular, to the unrealizable

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correlation functions. It is possible for this purpose to use also the method of coordinate-by-coordinate descent, but this method is more convenient for the quantified FM signals and it will be examined below.

## Chapter 10.

## SYNTHESIS OF QUANTIFIED FM SIGNALS WITH GOOD CORRELATION PROPERTIES.

## 10.1. Use/application of a criterion of proximity.

In §9.1 it was established that KFM signals relate to the composite/compound ones. The spectrum of the code of KFM signal has a value

$$H(\omega) = \sum_{i=1}^n x_i e^{-j\omega i} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \lambda_i e^{-j\omega i} \quad (10.1)$$

Here  $n$  - number of samples; value  $\lambda_i$  characterize their signs and allow/assume values of  $\pm 1$ . The latter is a main difference of KFM signals from the composite/compound signals of other types.

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We saw also that most important problem is finding KFM signals with the low remainders/residues of correlation function, i.e., approximation/approach to a correlation function of single the sample. Specifically, this task is examined in this chapter<sup>1</sup>.

FOOTNOTE <sup>1</sup>. The fundamental results of this were main published by

the authors in article [10]. The asymptotic solution, close to our, previously obtained by L. Ye. Varakin [14]. ENDFOOTNOTE.

The method of its solution, applied to the arbitrary composite/compound signals, is formulated in §7.4. Method assumes the best quadratic approximation of the amplitude spectra: the spectrum of the unknown KFM signal and the spectrum of single sample - in accordance with general/common/total criterion (7.17). As usual, matter is reduced to the maximization of the coefficient of proximity  $C(x, y)$  moreover in accordance with (7.44) for KFM signals we have

$$C(x, y) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \lambda_i y_i. \quad (10.2)$$

Here values  $y_i$  depend on the phase spectrum of the generating signal

$$y_i = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\alpha(\omega)} e^{-j\omega i} d\omega = \frac{1}{\pi} \int_0^{\pi} \cos[\alpha(\omega) - i\omega] d\omega. \quad (10.3)$$

Phase spectrum  $\alpha(\omega)$  is arbitrary, but, as for the not quantized FM signals, should be been bounded the odd phase spectra

$$\alpha(-\omega) = -\alpha(\omega); \quad \alpha(0) = 0. \quad (10.4)$$

This odd parity is taken into consideration in (10.2) and (10.3).

The task in question is, thus, of finding of phase spectrum  $\alpha(\omega)$  and values  $\lambda_i$  (equal to  $\pm 1$ ), with which the coefficient of proximity (10.2) attains maximum. This corresponds to the overall diagram of the use/application of a hypothesis of the proximity:

changes  $\lambda_i$  indicate displacement/movement over a permissible multitude of KPM signals, and changes  $a(\omega)$  - on a desired multitude of signals with the assigned correlation function (corresponding to single sample).

As in other similar tasks, the maximization of the coefficient of proximity can be fulfilled in any order. We will function as follows.

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First, fixing/recording phase spectrum  $\alpha(\omega)$  and, therefore, value  $y_i$ , let us find signs  $\lambda_i$  with which the coefficient of proximity (10.2) it is maximum, and then let us fulfill maximization also on  $\alpha(\omega)$ . The first stage is the task of approximation (design) on a permissible multitude of KPM signals, analogous of that examined in §9.4.

If values  $\lambda_i$  allow/assume only values of  $\pm 1$ , and  $y_i$  are fixed/recorded, then, as it is clear from (10.2),

$$C(x, y) \leq \frac{1}{\sqrt{n}} \sum_{i=1}^n |y_i|.$$

Here is reached equality, only if

$$\lambda_i = \text{sign } y_i. \quad (10.5)$$

i.e. when signs  $\lambda_i$  and  $y_i$  coincide. This determines optimum values

$\lambda_i$ , ensuring best approximation on the permissible set  $X$ , moreover

$$C(X, y) = \max_{x \in X} C(x, y) = \frac{1}{\sqrt{n}} \sum_{i=1}^n |y_i|. \quad (10.6)$$

This result is completely analogous to theorem of §9.4. It is now necessary to find the phase spectrum  $\alpha(\omega)$ , for which the coefficient of proximity (10.6) is maximum. The signs of the samples of optimum KFM signal are determined then according to (10.5).

## 10.2. Asymptotic solution.

The optimization of phase spectrum we will fulfill under the assumption of a large number of samples  $n$ , when to integral (10.3) it is possible to use the method of steady state (8.24). Analogous with that presented in §9.7 we obtain

$$y_i \approx \begin{cases} \sqrt{\frac{2}{\pi \alpha''(\omega_i)}} \cos \left[ \Phi(\omega_i) + \frac{\pi}{4} \right] & \text{при } 0 < \omega_i < \pi; \\ \frac{1}{2} \sqrt{\frac{2}{\pi \alpha''(\omega_i)}} \cos \left[ \Phi(\omega_i) + \frac{\pi}{4} \right] & \text{при } \omega_i = 0; \pi; \\ 0 & \text{при } 0 > \omega_i > \pi. \end{cases} \quad (10.7)$$

Key: (1). with.

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Here  $\omega_i$  - the point of steady state, determined from the equation

$$\alpha'(\omega_i) = i; \quad i = 1; 2; \dots n. \quad (10.8)$$

and function  $\Phi(\omega)$  has a value

$$\Phi(\omega) = \alpha(\omega) - \omega\alpha'(\omega) = \alpha(0) - \int_0^{\omega} \omega\alpha''(\omega) d\omega = - \int_0^{\omega} \omega\alpha''(\omega) d\omega. \quad (10.9)$$

Let us note that the approximation/approach of steady state has an error in order  $1/\sqrt{n}$ . Its use/application is limited also by the cases when derivative  $\alpha'(\omega)$  vary monotonically in the interval of integration  $(0, \pi)$ . It is concrete/specific/actual, in (10.7) we assume  $\alpha'(\omega)$  increasing, so that  $\alpha''(\omega) > 0$ . This limitation restricts the class of the signals in question, and we already indicated that it can, generally speaking, lead to the loss of the best signals. This question additionally is discussed below.

But use/application of a method of steady state gives direct analytical dependence  $y$  on  $\alpha(\omega)$ , and this makes it possible to fulfill research for the maximum to the end/lead. Substituting (10.7) in (10.6), it is possible with an error in the order  $1/n$  (smaller than an error in the method of steady state) to replace sum with integral. As a result it is obtained

$$C(X, y) = \sqrt{\frac{2}{\pi n}} \int_0^{\pi} \sqrt{\alpha''(\omega)} \left| \cos \left[ \Phi(\omega) + \frac{\pi}{4} \right] \right| d\omega + \\ + O\left(\frac{1}{\sqrt{n}}\right). \quad (10.10)$$

Is here carried out also the replacement of the variable/alternating of integration in accordance with (10.8). Correction term considers an error in the method of steady state.

The obtained relationship/ratio is completely analogous (with 9.27) and further research repeats conclusion/output of §9.7. After using expansion (9.29), we negligible rapidly-vibrating components/terms/addends under the integral (as was shown in §9.9, this is connected with the further error, which is also of the order  $1/\sqrt{n}$ ) and we come as a result to the maximization of value

$$C(X, y) \approx \frac{2}{\pi} \sqrt{\frac{2}{\pi n}} \int_{\omega_1}^{\omega_n} \sqrt{x''(\omega)} d\omega.$$

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The latter is implemented with the help of the Schwarz-Buniakowski inequality upon consideration of condition (10.8):

$$C^2(X, y) \leq \frac{8}{\pi^2 n} \int_{\omega_1}^{\omega_n} x''(\omega) d\omega \int_{\omega_1}^{\omega_n} d\omega = \frac{8}{\pi^2 n} [x'(\omega_n) - x'(\omega_1)] [\omega_n - \omega_1] = \frac{8}{\pi^2} \frac{n-1}{n} \frac{\omega_n - \omega_1}{\pi} \leq \frac{8}{\pi^2} \frac{n-1}{n}.$$

Here is reached equality, only if

$$\omega_1 = 0, \omega_n = \pi \text{ и } x''(\omega) = \frac{n-1}{n} = \text{const} \text{ при } 0 \leq \omega < \pi. \quad (10.11)$$

Key: (1). with.

These relationships/ratios determine optimum phase spectrum in the asymptotic approximation/approach of a large number of samples.

Coefficients  $y_i$  are found further from (10.7) - (10.9)

$$\begin{aligned}\omega_i &= \pi \frac{i-1}{n-1}; \\ \Phi(\omega_i) &= -\frac{n-1}{\pi} \frac{\omega_i^2}{2} = -\frac{\pi}{2} \frac{(i-1)^2}{n-1}; \\ y_i &= \sqrt{\frac{2}{n-1}} \cos \left[ \frac{\pi}{2} \frac{(i-1)^2}{n-1} - \frac{\pi}{4} \right], \quad 1 \leq i \leq n.\end{aligned}$$

Finally, the signs of the samples of the unknown KFM signal are determined according to (10.6)

$$\lambda_i = \text{sign} \cos \left[ \frac{\pi}{2} \frac{(i-1)^2}{n-1} - \frac{\pi}{4} \right], \quad 1 \leq i \leq n. \quad (10.12)$$

The maximum coefficient of proximity, attained in the asymptotic approximation/approach, it comprises

$$C(X, Y) = \frac{2\sqrt{2}}{\pi} \sqrt{\frac{n-1}{n}} + o\left(\frac{1}{\sqrt{n}}\right) = \frac{2\sqrt{2}}{\pi} + o\left(\frac{1}{\sqrt{n}}\right), \quad (10.13)$$

which completely will be coordinated with the results of the previous chapter.

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### 10.3. Iterative refinements by coordinate-by-coordinate descent.

The approximate, asymptotic solution, found above, can be made more precise, applying iterative methods. For example, with the help of the successive design it is possible, analogous with the case of the nonquantized signals, to take into account the error in the method of steady state (Fresnel pulsations), and also other inaccuracies in the previous calculation. Algorithm of these refinements even somewhat simpler than in §9.6. But, apparently, to a question about the refinements here one should approach from somewhat different positions.

The previous solution, based on the criterion of proximity, assumes approximation/approach to the amplitude spectrum of single sample, namely: we seek the KFM signal whose amplitude spectrum  $b_x(\omega)$  satisfies condition, see (7.17)

$$d^2(x, Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \left| \frac{\sin \omega/2}{\omega/2} \right| - b_x(\omega) \right]^2 d\omega = \min.$$

This criterion, although it is connected somehow with the appropriate

correlation functions, nevertheless does not guarantee their best approximation, in particular, the low level of remainders/residues. This confirms concrete/specific/actual calculations. The remainders/residues of correlation functions for the KFM signals, found from formula (10.12), noticeably exceed minimally known ones and during the large compression they reach approximately/exemplarily  $1.5/\sqrt{n}$ .

FOOTNOTE <sup>1</sup>. For  $n=13$  formula (10.12) gives Barker's signal with the smallest possible remainders/residues. ENDFOOTNOTE.

To preferably fulfill refinement, using the more straight/more direct criteria, connected directly with the remainders/residues of correlation function. We will use as a measure of the quality of signal the maximum remainder/residue of the correlation function

$$\mu = \max |R_k|, 1 \leq k < n. \quad (10.14)$$

the sum of the squares of all remainders/residues

$$\Delta_2 = \sum_{k=1}^n R_k^2, \quad (10.15)$$

and also the sum of their fourth powers

$$\Delta_4 = \sum_{k=1}^n R_k^4. \quad (10.16)$$

Two first criteria extensively are used with the synthesis, the role of the latter is clarified further.

All criteria indicated are some functionals (or functions) from the parameters of signal  $\lambda$ . Should be selected the adequate/approaching iterative method for their minimization. During this selection it is necessary to take into account that, in the first place, the arguments  $\lambda$  allow/assume only values of +1 and -1, and, in the second place, method must be sufficient to economical ones so that the calculations would prove to be virtually feasible with a large number of samples. We used under these conditions the method of the coordinate-by-coordinate descent which, in general, consists of the following.

Let us assume it is necessary to minimize function  $f(\lambda_1, \lambda_2, \dots, \lambda_n)$ , depending on the  $n$  arguments (coordinates). Being transmitted from certain initial approximation/approach

$$\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_n^{(0)}).$$

we attempt to change the value, for example, of the first argument, after assuming

$$\lambda_1^{(1)} = \lambda_1^{(0)} + \alpha.$$

where  $\alpha$  - selected previously space of iterations. If this gives the

decrease of the function being investigated, is accepted new value, i.e.,  $\lambda_1^{(0)}$  it is substituted on  $\lambda_1^{(1)}$ . Otherwise is done the attempt to be shifted in the opposite direction, after assuming  $\lambda_1^{(1)}$ :

$$\lambda_1^{(1)} = \lambda_1^{(0)} - \alpha.$$

Even if this attempt is unsuccessful, is leaved previous value  $\lambda_1^{(0)}$ .

Then is implemented analogous displacement on another coordinate, for example,  $\lambda_2$ .

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Having selected all coordinates and after changing them in the directions, which lead to the desired decrease of function, we obtain the first approximation

$$\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_n^{(1)}).$$

Then process is repeated.

After such several stages the approximations/approaches cease, since changes in each of the coordinates do not give the desired decrease of function. This can occur not only the unknown minimum, but also due to the high value of space  $\alpha$ . Therefore after the stop of iterations the space they reduce, for example doubly, and they again attain improvement, changing all coordinates alternately. Finally iterations cease, when there is no improvement even with the

sufficiently low pitch.

The method of coordinate-by-coordinate descent does not require the calculation of derivatives, in connection with which it is less labor-consuming than gradient. This method ensures also the higher speed of convergence and is insensitive to ravining of functional [52]. Moreover, the presence of "ravines" can be used, if to use the appropriate modification of coordinate-by-coordinate method, which ensures increasing motion in the direction, which approximately/exemplarily corresponds to the low place of ravine [78].

A deficiency/lack in the coordinate-by-coordinate descent lies in the fact that after catching accurately into the "bottom" of ravine, it is possible not to be shifted on the low place, if displacement on each of the coordinate directions is connected with the lift to the "slopes" [78]. In other words, are possible the false points of stop, which do not coincide with the minimum of function. Besides this, the point into which it gives coordinate-by-coordinate descent, generally speaking, depends on the order of sorting coordinates. Finally, as with other similar methods, we come in general, to the local, but not global minimum of function.

In the problem of the synthesis of KPM signals the method of

coordinate-by-coordinate descent additionally is simplified because the parameters (coordinates)  $\lambda_i$  allow/assume only values of +1 and -1. Therefore each coordinate can be changed in the unique direction so that the sign  $\lambda_i$  would vary for the reverse/inverse. Is assigned also the length of space  $|\alpha|=2$ .

As a result, coordinate-by-coordinate descent is reduced in our problem to the following. After taking one of the coordinates  $\lambda_i$  we attempt to change its value with +1 to -1 or vice versa.

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If this gives the decrease of the functional ( $\mu$ ,  $\Delta_2$  or  $\Delta_4$ ) being investigated, is accepted new value, otherwise is leaved old. In each stage this is made with all coordinates  $\lambda_i$  from  $i=1$  to  $i=n$ .

FOOTNOTE 1. In order to decrease the effect of the order of sorting, we in each case implemented descent twice - in the ascending order and with decrease of number  $i$ . Usually the results were identical.  
ENDFOOTNOTE.

Iterations cease, when change in any of the coordinates does not give the desired decrease of functional.

With the fulfillment of iterations on criteria (10.14) - (10.16) it is necessary to repeatedly compute the values of correlation function<sup>2</sup>

$$R_k = \sum_{i=1}^{n-k} \lambda_i \lambda_{i+k}; \quad k = 1, 2, \dots, n-1. \quad (10.17)$$

FOOTNOTE 2. As in §9.2, we use the further nonstandardized correlation functions for which  $R_0 = R(0) = n$ . ENDFOOTNOTE.

With a large number of samples straight/direct calculation according to this formula is very labor-consuming: it is required order  $n$  of operations in order to find one value  $R_k$  and order  $n^2$  of operations in order to obtain all values of correlation function. However, with sign change one sample it is possible not to compute anew correlation function, but to supplement to its previous values of the corrections, computed from the formula

$$\Delta R_k = \begin{cases} -2\lambda_m(\lambda_{m+k} + \lambda_{m-k}) & \text{при } m-k \geq 1, m+k \leq n-1; \\ -2\lambda_m\lambda_{m+k} & \text{при } m-k < 1, m+k \leq n-1; \\ -2\lambda_m\lambda_{m-k} & \text{при } m-k \geq 1, m+k > n-1. \end{cases}$$

Key: (1). with.

Here  $\lambda_m$  - old value of that varied sample. The use of this correction significantly reduces calculations and considerably widens practical possibilities. It is now required only order  $n$  of operations for obtaining all values  $R_k$ . As a result the machine time,

expended on the iteration, is shortened approximately/exemplarily proportional to a number of discretized signal.

#### 10.4. Results of synthesis.

The results of the synthesis of KFM signals with a number of samples  $n$  from 13 to 901 are given in Table 10.1. In the first column is indicated a number of samples.

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504

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Table 10.1.

(1) число дисков	(2) Начальное приближение		(3) Спуск по квадратиче- скому критерию		(4) Спуск по минимакс- ному критерию		(5) Спуск по среднестепен- ному критерию		(6) Известный сигнал	(7) Спуск слу- чайного начального приближе- ния
	$\Delta_0$	$\mu$	$\Delta_0$	$\mu$	$\Delta_0$	$\mu$	$\Delta_0$	$\mu$	$\mu$	$\mu$
13	6	1	6	1					1	
19	97	7	49	3	49	3	41	3	3	3
23	67	5	59	3	59	3	67	4	3	3
31	119	5	111	9	147	4	132	4	3	3
37	138	5	138	5	218	5	154	5	4	
41	308	9	116	5	388	5	280	6	4	
43	463	13	269	6	281	5	301	5	4	4
47		11	199	7	611	7	183	4	4	
53	362	9	286	6	566	7	315	5	5	
59	693	11	413	9	717	7	550	8		
61	446	11	358	7	806	7	499	7	5	5
63	431	9	343	7	611	6	495	7	6	6
67	1009	13	505	7	1121	8	685	7	5	
71	755	9	483	7	1159	8	707	9	5	
73	636	11	538	8	1088	8	604	7	6	
79	1079	13	715	7	1531	9	976	8		
91	2333	15	1237	10	1517	8	1313	9		
93	886	11	702	13	1694	9	870	8	6-8	
95	1255	17	871	9	1935	9	1227	8		
97	2392	19	1364	9		9	1277	10	7	7
99	1217	13	1033	13	2041	9	1065	9		
101	1378		1126	8			1599	9		
103	1947	15	1011	11	2005	9		9	6-8	
105	1876	13	1285	13	2056	10	1508	9		
107	1965	17	1361	11		11	1405	9		

## Tabelle 10.1 Cont'd.

109	1486		1294	13			1418	9	
111	1506	13	1275	13	2167	9	1495	10	7-8
113	2168	15	1152	9		11	1828	9	
115	2601		1117	10			1569	9	7
117	2170		1342	11			1782	12	
119	2299	15	1571	11		11	2307	10	
121	2076	17	1704	12	3736	12	2220	11	7-8
123	2613	21	1685	14	3809	12	2193	10	
125	2294	19	2070	11	2918	10	1602	7	
251	9349	27	7393	17	14701	16	7681	13	
253	8222	20	6674	21	16602	17	8890	13	11-12
255	9431	23	7023	16		17	9199	15	
257	12832	31	7064	18	16036	18	10168	15	
259	13777	41	8833	17	15733	17	8701	13	
261	6642	31	6386	27	20462	19	8510	15	
299	19053	37	10073	18	22169	20		17	
301	13702	29	9422	19	22394	20	11210	16	13
303	12151	29	9367	23	21343	18	11767	15	
305	13672		9452	19			12614	16	
503	36779	47	25235	27		25	35107	22	
505	42812	51	28244	33	65472	26	35836	21	18
511	58175	55	27311	28	70007	27	39427	23	
513	37632		26320	25			39944	21	
901	119194	59	83510	47		40	100154	26	

Key: (1). Number of samples. (2). Initial approximation/approach.

(3). Descent along quadratic criterion. (4). Descent according to

minimax criterion. (5). Descent along mean-exponential criterion.  
(6). Known signal. (7). Descent of random initial  
approximation/approach.

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In the following two columns are given the values  $\Delta_2$  and  $\mu$  for the initial approximation/approach, obtained according to asymptotic formula (10.12). Further are respectively arranged/located the results of descent along the rms (10.15), the minimax (10.14) and mean-exponential (10.16) to criteria. The tenth column shows the maximum remainders/residues of the best known signals, found with other methods (see §9.2).

The comparison of data of table makes it possible to consider that the method of synthesis examined gives sufficiently good results. On the level of maximum remainder/residue the obtained signals are only a little inferior to the best codes, known earlier, and by the obtained path of vast sorting.

Is focused attention, that with the descent along the root-mean-square criterion value  $\mu$  noticeably grows with an increase in the number of samples and for  $n=901$  it reaches  $1.51\sqrt{n}$ . This can be explained by the fact that with large  $n$  the quadratic criterion

weakly reacts to the separate large overshoots. With the descent along the minimax criterion the picture is reverse/inverse - criterion reacts to the maximum surges  $\mu$ , and value  $\Delta_2$  sharply grows (Table 10.1).

The analysis of the best ones of the obtained by us codes shows that with the low maximum remainder/residue they possess a comparatively low sum of squares  $\Delta_2$ . The examination of detection problem shows, besides the fact that they are important both the maximum remainders/residues and rms level. Depending on situation, in particular from the relationship/ratio of the useful and interfering signals, the dominant role plays either that or another criterion [15]. Therefore it was desirably use this criterion of the synthesis which would react not only to the maximum remainders/residues, but also to their total level. This led us to mean-exponential criterion (10.16), which corresponds to approximations/approaches in space  $L^4$ . As can be seen from table, such approximations/approaches lead usually to the best results, in particular, the value of maximum remainder/residue comprises  $(0.7-0.9) \sqrt{n}$  1.

FOOTNOTE 1. It is possible to assume that for even larger  $n$  an expediently further increase in the degree, for example approximation/approach in spaces  $L^6$  or  $L^8$ . ENLFOOTNOTE.

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Table 10.2.

п	к	(1) Код сигнала
16	2	0000111010100100
17	2	11100000010100110
19	3	1101101110111000111
23	3	11010010001000100000111
31	3	10100101111100001100110 11101111
43	4	10100010111000000100100 10001110011101001010
47	4	111111110000111100011 001100110110110100101 01010
53	5	00111111000000110000111 00111011011011011011100 1010101
61	5	110000100010110000101100 110001111111010110000011 1011010111001
97	7	1101111110111001110111 110101100010010101011011 011100000101100001011100 0110101111010001010011111 0010
125	7	00011111111111000100011 11000001111000111000111 00011101110011001100110 110011011011011011011010 110101101011010100101010 1010110
251	13	10011101111111011000000 00000111111000001111111 100000111110001111000000 10000111001111000111000 1100111000110011000 010011000100110010001100 110110011011001111101101 101101101101101101101000 011010010100101001010010 101101010101101011101001 01010101010101010

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303	15	000001010101010101010101 01010101011010101001010 100101011010010101101011 010010011100100101101101 001001001001001101101101 100100110010011011001100 100011001101110011001100 110000110011001110011100 111000111001111000111001 011100011111000111110000 111111000001111110000000 011111111000100000001111 11111111111111111111
513	21	00100011001010100010 10101001000101010101 01001000101011010101 01101010100101011010 10101101001010000101 10101101010010010100 10110100100101101011 01001001001011011011 01111101001101001101 10110010110110010010 01101100100110110011 01100110010011001100 10011000100110011001 10011001100011001100 10000110001000010011 10011100111001110001 10001100011100011101 01110001110000111001 11100001111000111100 01111100001110101011 11100000111110000011 1110000001111110000 00011110111000000001 1111111111000000100 0000011111111111111 1011111111111111
901	25	01011101101101111010111010111000010 00101010101010101110100101011101010 10010111010110101010100101010101101 0101010010110101010110000001101011 01010110101101010110100101010110110 10100101011010010101101011001010010 10010100100101001001011011010110100

901

26

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10010010100110100101101101101000101
01010100101101101101101101101100
10010110010111011010011001001001101
00011011011001010101100100110010011
00100110010011001101100111011000001
10010001001100110011001100110011001
11011001100110011001010011001100011
00110011100110011100110011001110011
10001110011000111001100011000110001
11001110001110011100001100011000110
00011100011100011010111100001110000
11100011110001111000111100001110000
1111000111100001111000011111000011
111100011110000011110000011111000
0001111100000011111100000011111100
0000010111110100000001111111000000
00011111111101000000000111111101
1111000100000000001100111111110111
11111111111111111110111111

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Key: (1). Code of signal.

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Asymptotic initial approximation/approach (10.12) determines the signals, which possess the property of monotonicity, the frequency of commutations gradually increases toward the end of the signal. This monotonicity, caused by the limitations of the previous conclusion/output, one way or another is retained also after iterations. It was possible to assume that this, insufficiently general/common/total structure of initial approximation/approach did not permit us to obtain smallest possible remainders/residues.

For explaining this question was made the following calculation. Was implemented descent along rms criterion (10.15), but as the initial approximation/approach were used the random codes, obtained via the equiprobable selection of signs. Initial approximation/approach repeatedly was changed for each  $n$ . Results are given in the latter/last column of table. In many instances actually/really were obtained minimum known remainders/residues. In Table 10.2 gives the best found by us signals.

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Appendix 1.

# PROPERTIES OF SPHEROIDAL FUNCTIONS.

Spheroidal functions  $\psi_n(\xi)$  are the eigenfunctions of the integral equation

$$\int_{-1}^1 \psi_n(\xi') G(\xi, \xi') d\xi' = \lambda_n \psi_n(\xi) \quad (1)$$

with the kernel

$$G(\xi, \xi') = \frac{\sin c(\xi - \xi')}{\pi(\xi - \xi')} = \frac{c}{2\pi} \int_{-1}^1 e^{-j(\xi - \xi')\eta} d\eta. \quad (2)$$

They possess the series/row of the properties, which are of interest for the theory of signals. Let us point out briefly these properties, relying on the series/row of sources [43, 65-67, 80]. The translations/conversions of fundamental works on spheroidal functions are in [95].

1. System of spheroidal functions. As it follows from (2.9), the quadratic form

$$\int_{-1}^1 \int_{-1}^1 s(\xi') s^*(\xi) G(\xi, \xi') d\xi d\xi'$$

is partial energy of signal  $s(\xi)$ , included in the band  $(-c, c)$ . This value is positive for any function  $s(\xi)$ . The aforesaid means that the symmetrical kernel  $G(\xi, \xi')$  is determined positively. From the theory of integral equations it is known that under these conditions eigenfunctions  $\psi_n(\xi)$  form complete orthogonal system in interval  $(-1, 1)$ .

Equation (1) determines functions  $\psi_n(\xi)$  with an accuracy to the arbitrary normalizing factor. Therefore it is possible to carry out such standardization that<sup>1</sup>

$$\int_{-1}^1 \psi_n(\xi) \psi_m^*(\xi) d\xi = \begin{cases} 0 & m \neq n; \\ 1 & m = n. \end{cases} \quad (3)$$

Key: (1). with.

FOOTNOTE 1. In the works on spherical functions frequently are used other rules of standardization. Standardization (3) is used, in particular, in [65, 68]. ENDFCCTNCTE.

Any function  $w(\xi)$ , integrable squared, can be in interval  $(-1, 1)$  expanded in the convergent (on the average) Fourier series

$$w(\xi) = \sum_{n=0}^{\infty} a_n \psi_n(\xi). \quad (4)$$

where coefficients  $a_n$  are determined in the form

$$a_n = \int_{-1}^1 \varphi_n(\xi) \psi_n^*(\xi) d\xi. \quad (5)$$

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Since kernel (2) is real, spheroidal function  $\psi_n(\xi)$  is also real; therefore the sign of composite coupling in formulas (3) and (5) it is possible not to write out.

It is not difficult to show also (see for example [66]), what function  $\psi_n(\xi)$  is even for even  $n$  and is odd - for odd  $n$ .

2. Fourier transform from  $\psi_n(\xi)$ . Let us compute Fourier integral of  $\psi_n(\xi)$ , which let us register in the form

$$\varphi_n(\eta) = \int_{-1}^1 \psi_n(\xi) e^{jc\eta\xi} d\xi. \quad (6)$$

Taking into account (1) and (2), we find:

$$\begin{aligned} \varphi_n(\eta) &= \int_{-1}^1 e^{jc\eta\xi} d\xi \frac{1}{\lambda_n} \int_{-1}^1 \psi_n(\xi') G(\xi, \xi') d\xi' = \\ &= \frac{1}{\lambda_n} \int_{-1}^1 d\eta' \int_{-1}^1 \psi_n(\xi') e^{jc\xi'\eta'} d\xi' \frac{c}{2\pi} \int_{-1}^1 e^{-jc(\eta-\eta')\xi} d\xi = \\ &= \frac{1}{\lambda_n} \int_{-1}^1 \varphi_n(\eta') G(\eta, \eta') d\eta'; \end{aligned}$$

here

$$G(\eta, \eta') = \frac{c}{2\pi} \int_{-1}^1 e^{jc(\eta-\eta')\xi} d\xi = \frac{\sin c(\eta-\eta')}{\pi(\eta-\eta')}.$$

This result means that the Fourier transform  $\varphi_n(\eta)$  satisfies the same integral equation

$$\int_{-1}^1 \varphi_n(\eta') G(\eta, \eta') d\eta' = \lambda \varphi_n(\eta).$$

as function itself  $\psi_n(\xi)$ . Consequently, spheroidal function and its Fourier transform are characterized by only scale factor. In other words, we come to the relationship/ratio

$$\int_{-1}^1 \psi_n(\xi) e^{jc\eta\xi} d\xi = a_n \psi_n(\eta), \quad (7)$$

which is also the integral equation, which are determining spheroidal functions.

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Eigenvalues  $a_n$  of this equation can be connected with eigenvalues  $\lambda_n$  of initial equation (1). Actually/really, taking into account that  $\varphi_n(\xi)$  is real, we obtain via the iteration

$$\begin{aligned} \psi_n(\eta) &= \frac{1}{a_n} \int_{-1}^1 \psi_n(\xi) e^{jc\eta\xi} d\xi = \\ &= \frac{1}{a_n} \int_{-1}^1 e^{jc\eta\xi} d\xi \frac{1}{a_n^*} \int_{-1}^1 \psi_n(\eta') e^{-jc\xi\eta'} d\eta' = \\ &= \frac{1}{|a_n|^2} \frac{2\pi}{c} \int_{-1}^1 \psi_n(\eta') G(\eta, \eta') d\eta'. \end{aligned}$$

Thus, equation (1) can be considered as first iteration (7), if we assume  $|x_n|^2 = 2\pi\lambda_n/c$ .

It is not difficult to show also that value  $\alpha_n$  is real, if  $\psi_n(\xi)$  is even, and imaginary, if  $\psi_n(\xi)$  - is odd [66].

These considerations lead to the dependence

$$\alpha_n = j^n \sqrt{\frac{2\pi\lambda_n}{c}},$$

taking into account to which formula (7) can be rewritten in the form

$$\int_{-1}^1 \psi_n(\xi) e^{j\eta\xi} d\xi = j^n \sqrt{\frac{2\pi\lambda_n}{c}} \psi_n(\eta/c). \quad (8)$$

Inverse transformation of Fourier gives

$$\int_{-\infty}^{\infty} \psi_n(\eta) e^{-jk\eta} d\eta = \begin{cases} j^{-n} \sqrt{\frac{2\pi}{c\lambda_n}} \psi_n(\xi/c) & \text{if } -c \leq \xi < c, \\ 0 & \text{if } |\xi| > c. \end{cases} \quad (9)$$

Key: (1). with.

We initially were interested in the behavior of spheroidal functions in limited interval  $(-1, 1)$ . But formulas (8), (9), obviously, determine these functions on the entire axis  $(-\infty, \infty)$ . From (9) it follows that function  $\psi_n(\eta)$ , examined/considered on the entire axis, has a Fourier transform of the limited extent. In that region where this conversion is excellent from zero, it repeats (without taking into account constant factor) function itself. This it indicates certain generality of spheroidal functions and functions of

Hermite (in particular, Gaussian signal). The latter possess a similar duality and are close to the spheroidal functions with  $c \rightarrow \infty$ . This more fully question is traced in connection with asymptotic expansions of spheroidal functions [65].

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3. Double orthogonality. Above it was shown that the set of functions  $\psi_n$  was orthogonal in interval  $(-1, 1)$ . This is the direct consequence of that fact that the spheroidal functions are the solutions of homogeneous equation (1) with the positively determined symmetrical kernel. The rare special feature/peculiarity of spheroidal functions consists in the fact that, besides orthogonality in the finite interval, these functions are orthogonal also in the interval  $(-\infty, \infty)$ . Actually/really, taking into account (9), on the basis of equality Parseval we can register

$$\int_{-\infty}^{\infty} \psi_n(\eta) \psi_m^*(\eta) d\eta = \frac{j^{m-n}}{c \sqrt{\lambda_n \lambda_m}} \int_{-c}^c \psi_n(\xi, c) \psi_m^*(\xi, c) d\xi.$$

The obvious replacement of variable/alternating leads right side to form (3), and is obtained

$$\int_{-\infty}^{\infty} \psi_n(\eta) \psi_m^*(\eta) d\eta = \begin{cases} 0 & \text{при } m \neq n, \\ 1/\lambda_n & \text{при } m = n. \end{cases} \quad (10)$$

Key: (1). with.

The double orthogonality of spheroidal functions makes from with ideal apparatus with the solution of such problems of the theory of signals as the approximation of arbitrary signal with the help of the function whose spectrum is limited by extent, or the extrapolation of the signals of the limited band out of the given one time interval [7, 67].

Let us emphasize, however, that if in interval  $(-1, 1)$  the system of spheroidal functions is complete, then for the infinite interval this not then. Actually/really, since functions  $\psi_n(\eta)$  have a spectrum of the limited extent (this is clear from (9), their superposition cannot form the arbitrary signal whose spectrum falls outside band  $(-c, c)$ .

However, it is easy to show that in the class of signals with the spectrum, limited by the band indicated, the system of spheroidal functions is complete, so that any signal of this type can be decomposed according to functions  $\psi_n(\eta)$ , and this resolution is useful for entire axis  $-\infty < \eta < \infty$ .

## Appendix 2.

Determination of the concepts of amplitude, phase and instantaneous signal frequency.

As it was noted in input chapter, one of basic concepts of the theory of signals is the analytical signal  $s(t)$ , formed from the real signal  $u(t)$  during the addition by its imaginary component,

$$s(t) = u(t) + jv(t), \quad (1)$$

moreover the latter is found from the conversion of Gilbert:

$$v(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{t - \tau} d\tau. \quad (2)$$

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This selection of imaginary component is connected with determining of the fundamental characteristics of signal - its amplitude envelope, phase and instantaneous frequency. It will be, shown below that only during the use of conversion of Gilbert the characteristics indicated will be coordinated with the completely obvious physical requirements, so that any another selection of imaginary component in (1) is excluded [94].

It is not difficult to comprehend that the observed ((for example on the oscillograph) real signal  $u(t)$  it is possible more or less arbitrarily to disengage to the amplitude and fluctuating factors, i.e., to present in the form

$$u(t) = A(t) \cos \varphi(t) = A(t) \cos [\omega_0 t + \Phi(t)]. \quad (3)$$

In other words, only the left side of equality (3), signal  $u(t)$ , is the physically observed value, and for the concrete definition of right side, for determining the amplitude  $A(t)$  and phase  $\varphi(t)$  it is required certain "conjecture", speculative interpretation of the observed phenomenon. This means that are possible different definitions of amplitude and phase, and, as it will be shown, this ambiguity is connected with the selection of one or the other imaginary part  $v(t)$  in (1).

1. Connection/communication with composite representation of signal. Let us consider composite signal (1) with the arbitrary imaginary part of  $v(t)$ . After rewriting (1) in the form

$$s(t) = \sqrt{u^2(t) + v^2(t)} \exp \left\{ j \arctg \frac{v(t)}{u(t)} \right\} = A(t) e^{j\varphi(t)}, \quad (4)$$

where

$$u(t) = A(t) \cos \varphi(t) \quad \text{and} \quad v(t) = A(t) \sin \varphi(t).$$

it is not difficult to note that separation  $u(t)$  to the interesting us factors actually/really it is possible to fulfill differently,

but, when  $v(t)$  with any form is selected, amplitude and phase are determined unambiguously:

$$A(t) = \sqrt{u^2(t) + v^2(t)}; \varphi(t) = \operatorname{arctg} \frac{v(t)}{u(t)}. \quad (5)$$

It is easy to be convinced also of the reverse/inverse: any separation  $u(t)$  to the factors of form (3) indicates certain concrete/specific/actual selection of the imaginary component  $v(t)$ . Actually/really, if  $A(t)$  and  $\varphi(t)$  are undertaken so that  $u = A \cos \varphi$ , then, after placing  $v = A \sin \varphi$ , we come to the composite signal  $s(t)$  in the form (4).

Thus, with the assigned real signal  $u(t)$  is one-to-one conformity between its amplitude and phase, on one hand, and imaginary component  $v(t)$  of composite signal - on the other hand. In order to unambiguously determine amplitude and phase (and also instantaneous frequency  $\omega_c(t) = d\varphi/dt$ ), it is necessary and it suffices to indicate the rule of the selection of the imaginary component  $v(t)$  on the real signal  $u(t)$ . In other words, it is necessary to indicate operator  $L$ , which realizes the conversion

$$v(t) = L[u(t)], \quad (6)$$

each operator generating one of the possible determinations of amplitude, phase and frequency, and their complete set corresponds to all possible determinations.

## 2. Physical conditions for selection of operator.

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However, not with any operator  $L(u)$  the generated by it concepts of amplitude and phase will be coordinated with the physical, engineering representations, not any operator can be therefore recognized as satisfactory. Let us formulate conditions to the amplitude and phase of signal, based only on physical considerations, but such, that the need for their fulfillment occurs sufficiently obvious.

1). Let us require so that to the small changes in the initial signal  $u(t)$  would correspond small changes in its amplitude  $A(t)$  and phases  $\phi(t)$  (latter, if  $A(t) \neq 0$ ).

Since conversions (5) are continuous, for this is required the continuity of operator (6). Further, in the space of continuous operators are differentiated operators they form everywhere dense set. Using this, it is possible to consider that operator (6) is differentiated, i.e.,

$$L(u + \delta u) = L(u) + L'(u)\delta u + O(\delta u). \quad (7)$$

Transition/junction from the continuous operator to that differentiated indicates, strictly speaking that we change values of

$A(t)$  and  $\phi(t)$ , but so that for any signal these values are changed arbitrarily little. It is obvious, this replacement is permitted.

In general,  $L'(u)$  is derivative of the unknown operator, and we require, thus, the existence of this derivative for any signal  $u(t)$ .

2). Let us require so that the phase (and, therefore, instantaneous frequency) would not depend on the power (norm) of signal with its constant/invariable form.

This means that with any positive constant  $k$  replacement  $u(t)$  on  $ku(t)$  must not lead to the change  $\phi(t)$ , i.e., taking into account (5)

$$\frac{L(ku)}{ku} = \frac{L(u)}{u}.$$

Hence it follows that operator  $L$  must be uniform of the first degree:

$$L(ku) = kL(u), \quad k > 0. \quad (8)$$

3). There is a unique class of the signals for which amplitude, phase and frequency they are known completely accurately. These are the strictly harmonic, monochromatic oscillations/vibrations

$$u(t) = A_0 \cos(\omega_0 t + \Phi_0), \quad (9)$$

in which  $A_0$  and  $\Phi_0$  - constant. Any attempt to determine amplitude and phase for other signals is a generalization of the corresponding concepts, known for the harmonic case.

Therefore let us require so that for the harmonic signals the

introduced concepts of amplitude and phase become knowns.

I.e., for signal (9) we must obtain

$$A(t) = A_0, \varphi(t) = \omega_0 t + \Phi_0.$$

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From (5) it follows that for this the harmonic signal must be transformed by the completely specific form, namely

$$L[\cos(\omega_0 t + \Phi_0)] = \sin(\omega_0 t + \Phi_0). \quad (10)$$

We will show further that the operator of Gilbert (2) is to all only satisfying conditions indicated, and therefore the corresponding concepts of amplitude, phase and frequency are singularly permissible.

3. Proof of uniqueness. The unique linear (additive) operator, which satisfies condition (10) at any frequency  $\omega_0$ , is the operator of Gilbert [93, page 159-161]. Therefore we will demonstrate the uniqueness of permissible conversion (2), if we establish that from conditions (7) and (8) follows also the linearity of operator  $L(u)$ .

After introducing in the space of signals certain base, it is possible to reduce the problem to the analysis of the transformation of the multidimensional vector  $U = \{u_1, u_2, \dots, u_n, \dots\}$  into

multidimensional vector  $v = \{v_1, v_2, \dots, v_n, \dots\}$ . Components  $u_i$  and  $v_i$  are coefficients of the expansion of signals in terms of the selected base system; strictly speaking, a number of such components infinitely, but virtually always it is possible to be bounded to finite expansions.

In general, the transformation of the vectors indicated is assigned by system of equations

$$v_j = f_j(u_1, u_2, \dots, u_n, \dots); j = 1, 2, \dots$$

where  $f_j$  - arbitrary functions of many variable/alternating. We should show that with satisfaction of conditions (7), (8) these functions are linear. The appropriate proof we will lead for functioning two variable/alternating, generalization to the multidimensional case is obvious.

Let function  $f(x, y)$  be uniform the first degree and has both particular derived at all values of  $x$  and  $y$  (these conditions correspond (7) (8)). Then, in view of differentiability at point  $x=y=0$ , we have

$$f(x, y) = f(0, 0) + ax + by + e(x, y). \quad (11)$$

Here  $a$  and  $b$  - corresponding partial derivatives, and function  $e(x, y)$  on any ray/beam  $y = \gamma x$  vanishes more rapidly than  $x$ , i.e.,

$$\lim_{x \rightarrow 0} \frac{e(x, \gamma x)}{x} = 0. \quad (12)$$

But  $f(x, y)$  is also uniform and, in view of the known Euler

formula,

$$f(x, y) = x'f_x(x, y) + y'f_y(x, y).$$

Therefore  $f(0, 0) = 0$ . Further, from (11) we have

$$e(x, y) = f(x, y) - (ax + by).$$

We see that,

$e(x, y)$  is a difference in two uniform functions and, therefore, itself is uniform. Therefore on the ray/beam

$$e(x, yx) = xe(1, y). \quad (13)$$

in question.

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Since  $e(1, y)$  does not depend on  $x$ , on (12) and (13) it follows that  $e(x, yx)$  is equal to zero with all  $x$ , on entire ray/beam. Finally, in view of the arbitrariness of selected ray/beam  $e(x, y)$  it is equal to zero identically and, according to (11),  $f(x, y)$  is linear. This completes proof<sup>1</sup>.

FOOTNOTE <sup>1</sup>. We were based higher by differentiability  $f(x, y)$  at the unique zero point. This can cause the doubt of the correctness of the formulation of the problem, since the point indicated corresponds to the signal of zero point energy, which is not of interest. But from previous it follows also that the uniform function, not

differentiated somewhere, is not differentiated also in zero (otherwise it is linear, i.e., it is differentiated everywhere). Therefore the noted contradiction only seeming. ENDFOOTNOTE.

Thus, the conditions of differentiability (7) and uniformity (8) lead to the linearity of operator  $L(u)$ , and then condition for harmonic signals (10) proves the uniqueness of Gilbert's operator (2).

4. Discussion of results. The use/application of transformation of Gilbert in the theory of signals is well known, and in many works of his property thoroughly was studied from that point of view in order to be convinced of the suitability of the corresponding concepts of amplitude, phase and frequency (for example, see, [30]).

In connection with narrow-band, quasi-harmonic signals all proceeds happily. But when the band of signal is commensurated with the medium frequency, the demonstrative character of envelope is lost. In particular, if  $u(t)$  - the rectangular radio pulse of sufficiently short duration, the envelope  $A(t)$  differ from rectangular and contain the "tails" of infinite extent. These "tails" are reduced with an increase in the carrier frequency when signal approaches harmonic, but this structure of envelope does not give demonstrative representation in the non-narrow-band case.

In connection with similar contradictions frequently they are voiced about the fact that for the broadband signals the corresponding concepts of amplitude and phase have only formal character. Thus, analyzing one characteristic example, Cramer and Leadbetter write [96, page 307] that obtained with the help of the transformation of Gilbert the envelope "has no sense from the point of view of the physical content of concept.... Although the mathematical determination of envelope unambiguously, it is necessary to be for careful ones with the physical interpretations, which relate to the broadband signals".

Hardly it is possible to agree with similar propositions. In fact, here it is possible only to say that in the broadband case the transformation of Gilbert does not lead to the demonstrative description of the signal through the amplitude and the phase. But for this signal, substantially different from the harmonic, demonstrative description through the amplitude and the phase it can and not exist: indeed during the construction of this description we always attempt to preserve the narrow-band model, inadequate to the case in question.

The essence of this problem in any way is not reduced to

obtaining of demonstrative representations. From a physical point of view is more important not to break the accepted by us condition, which are reduced to the continuity of the determined concepts, to the independence of phase of frequency from the amplitude (scale) of signal and to the agreement with the known determinations for the harmonic oscillations. In the confirmation let us consider some known methods of measurement, in which these conditions are disrupted.

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1) The spread method of measurement of frequency is based on the calculation of number of zeros, zero-level intersections per unit time. In this case the measured value of frequency, obviously, does not depend on signal amplitude. For the harmonic oscillation of the measurement method to be carried out correctly and it is very accurate. This means that the second and the third of the accepted by us conditions are implemented. But continuity condition here is not satisfied: it is not difficult to indicate such signals and such slight disturbances, that a number of intersections will be changed with jump, several zero will be supplemented or will vanish. For the quasi-harmonic signals these jumps are usually unessential. But for the broadband signals with a small number of zero such methods of measurement does lead to the uncontrollable errors and the series/row virtually is suitable.

2) Even grosser discontinuity is allowed in the method of measurement of frequency, based on the so-called structural properties of signals. One of these properties gives the known differential equation  $u''(t) + \omega^2 u(t) = 0$ , valid for the harmonic oscillations. Based on this equation, in [97] it is proposed to measure the instantaneous frequency, using the relationship/ratio

$$\omega_c = \sqrt{-\frac{u''(t)}{u(t)}} \quad (14)$$

Always it is possible to fit this slight disturbance of signal, which at certain moment/torque will be obtained by  $u(t) = 0$  at the finite value of second derivative  $u''(t)$ . At this moment the value of frequency according to formula (14) goes to infinity, i.e., continuity is not observed.

The absurdity of this method of measurement is almost obvious, since it is unsuitable not for what signals, except strictly sinusoidal ones. Actually/really, if signal is modulated in the amplitude,  $u(t) = A(t) \cos \omega_0 t$ , substituting in (14), we obtain

$$\omega_c = \sqrt{\omega_0^2 - \frac{A''(t)}{A(t)} + 2\omega_0 \frac{A'(t)}{A(t)} \operatorname{tg} \omega_0 t}$$

Analogously, for ChM signal  $u(t) = \cos(\omega_0 t + \Phi(t))$  we have

$$\omega_c = \sqrt{[\omega_0 - \Phi'(t)]^2 + \Phi''(t) \operatorname{tg}[\omega_0 t - \Phi(t)]}$$

With any, now conveniently slow changes in the amplitude or frequency this result is deprived of any sense: in each period of the carrier frequency the radicand varies from  $+$  to  $-$ , and it is not possible to extract root simply. Only with the harmonic signal, for which correctly initial equation, result of measurement is correct.

3). Tikhonov proposed the determination of signal amplitude envelope on the base of operator [98]

$$z(t) = L(u) = u'(t)/\omega_0 \quad (15)$$

For the harmonic oscillation of frequency  $\omega_0$  this operator gives the same as the transformation of Gilbert. Furthermore, linear operator (15) satisfies the continuity conditions and uniformity. But for the harmonic oscillation of any other frequency, different from  $\omega_0$ , condition (10) is broken, and this leads to the explicit contradiction.

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For example, signal amplitude envelope  $u(t) = \cos \omega_1 t$  obtains expression

$$\begin{aligned} A(t) &= \sqrt{\cos^2 \omega_1 t + \frac{\omega_1^2}{\omega_0^2} \sin^2 \omega_1 t} = \\ &= \sqrt{1 + \frac{1}{2} \left( \frac{\omega_1^2}{\omega_0^2} - 1 \right) - \frac{1}{2} \left( \frac{\omega_1^2}{\omega_0^2} - 1 \right) \cos 2\omega_1 t} \end{aligned}$$

Rapid changes in the envelope (with the frequency  $2\omega_1$ ) here occur even for the harmonic signals, what, obviously, must not be.

4). Let us point out also the example when is broken the condition of uniformity, independence of frequency and phase from the signal amplitude. This example gives the measurement of frequency or phase by the corresponding discriminator without the preliminary amplitude limitation. Then the result of measurement depends substantially on the amplitude that it does not make it possible to apply such meters.

Let us note, however, that with the broadband signals the limiters conduct to the noticeable distortions of phase and frequency, in connection with which it is expedient to pass to other methods of measurement [94].

These examples show that the disturbance/breakdown at least of one of our requirements can lead to essential contradictions. Results

are either clearly absurd or they will not be coordinated with those expected even for the narrow-band signals. On the other hand, the transformation of Gilbert, satisfying all requirements indicated, in fact does not lead to the contradictions, since certain inadequacy of the concepts of amplitude and phase for broadband signals is caused by their nature itself. Finally, since only the transformation of Gilbert satisfies all conditions accepted, we come to the single-valued determination of amplitude and phase, and we also introduce naturally the concept of analytical signal.

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#### REFERENCES

1. N. I. Akhiezer. Lectures on the theory of approximation. Publishing house "science", 1965.
2. L. D. Bakhrakh, V. I. Trontsiy. Displaced problems of the synthesis of antennas. "radio engineering and electronics", 1967, Vol. 12, No 3.
  3. Barker R. H. Group synchronizing of binary digital system «Communication theory», ed. by W. Jackson, London. Butterworth scientific publications, 1953.
  4. Bochner A. M. Binary pulse compression codes. IEEE Trans. 1967, v. IT-13, № 2.
  5. de Buda R. Stationary phase approximations of FM spectra. IEEE Trans. 1966, v. IT-12, № 3.
6. D. Ye. Vakman. Asymptotic methods in linear radio engineering. Publishing house "Soviet radio", 1962.
7. D. Ye. Vakman. Serrated signals and the uncertainty principle in the radar. Publishing house "Soviet radio", 1965.
8. D. Ye. Vakman. Regular method of synthesis of FM signals. Publishing house is "Soviet radio", 1967.
9. D. Ye. Vakman. Optimum signals, which maximize the partial

space of the body of uncertainty/indeterminacy. "Radio engineering and electronics", 1967, Vol. 12, No 8.

10. D. Ye. Vakman, R. M. Sedletskiy, I. Z. Shapiro. Synthesis of the quantified FM signals with good correlation properties. "Radio engineering and electronics", 1970, Vol. 15, No 4.

11. D. Ye. Vakman, I. Z. Shapiro. One property of integral functions and its application/appendix in the theory of signals. "Radio engineering and electronics", 1970, Vol. 15, No 8.

12. D. Ye. Vakman, S. Sh. Fetker. To a question about the selection of the optimum weight processing of signal. "Questions of radio electronics", surveys. General technical, 1968, No 25.

13. L. Ye. Varakin. Theory of serrated signals. Publishing house "Soviet radio", 1970.

14. L. Ye. Varakin. Synthesis of the phase-keyed signals. "Radio engineering and electronics", 1969, Vol. 14, No 6.

15. L. Ye. Varakin. On the criteria of the synthesis of serrated signals. "Proceedings of VUZ. Radio electronics, 1970, Vol. 13, No 2.

16. P. M. Woodward. Probability theory and information theory to uses/applications to the radar. Publishing house "Soviet radio", 1955.

17. Chalk J. The optimum pulse-shape for pulse communication. Proc. IEE, 1950, v. 87, No 1.

18. Cook C. E., Paolillo J. A pulse compression predistortion function for efficient sidelobe reduction in high-power radar. Proc. IEEE, 1961, v. 55, No 4.

19. B. M. Gerasim. On the optimum permission/resolution of radar objects. "Radio engineering and electronics", 1969, Vol. 14, No 9.

20. V. L. Goncharov. Theory of interpolation and approximation/approach of functions. State Technical Press, 1934.

21. I. S. Gradshteyn, I. M. Ryzhik. Table of integrals, sums, series/rows and products. Fizmatgiz, 1962.

22. M. S. Gurevich. Signal of the final duration, which contain the maximum portion of energy in the assigned band. "Radio engineering and electronics", 1956, T. 1 No 3.

23. M. S. Gurevich. Spectra of radio signals. Publishing house "connection/communication", 1963.

24. R. S. Guter et al. Elements/cells of the theory of functions. Fizmatgiz, 1963.

25. V. F. Dan'yakov. To the construction of optimum program in the linear system. "Automation and telemechanics", 1964, Vol. 25, No 1.

26. "Parts and elements/cells of radars", Vol. 1. Trans. from Engl.; edited by A. Ya. Breytbart. Publishing house "Soviet radio", 1952.

27. Dolph C. L. A current distribution for broadside arrays which optimizes the relationship between width and side-lobe level. Proc. IRE, 1946, v. 34, № 6

28. M. A. Yevgrafov. Asymptotic evaluations/estimates and integral functions. Fizmatgiz, 1962.

29. Fowle E. N. A method of designing FM pulse compression signals. IEEE Trans. 1964, v. IT-10, № 1

30. Franks L. E. Signal theory. Prentice Hall, Publ. 1969

31. S. I. Zukhovitskiy, I. I. Avdeyeva. Linear and convex programming. Publishing house "science", 1967.

32. I. M. Ivanov et al. On the existence of Barker codes. "Proceedings of high school", the ser. of Radiofizik, 1960, Vol. 3, No 5.

33. L. V. Kantorovich, G. E. Akilov. Functional analysis in the standardized/normalized spaces. Fizmatgiz, 1959.

34. E. I. Krupitskiy, T. N. Sergeyenko. Problem of the minimization of the functional of the error in the class of relay functions and its application/appendix to the synthesis of FM signals

and line-source antennas. "Radio engineering and electronics", 1970, Vol. 15, No 2.

35. Ch. Cook, M. Bernfeld. Radar signals. Publishing house "Soviet radio", 1971.

36. Key E. L., Fowle E. N., Haggarty R. D. A method of designing signals of large time-bandwidth product. IRE Intern. Conv. Record, pt. 4, 1961.

37. Kay D. N. Chirp system will have enough bandwidth to give a range profile of irregularities six inches apart. Electronic Design, 1969, v. 17, No 14.

38. J. R. Klauder. The design of radar signals having both high range resolution and high velocity resolution. Bell System Techn. Journ. of 1960, v. 39, No 4. ("Foreign radio electronics", 1961, No 1).

39. J. R. Klauder, A. C. Price, S. Darlington, S. Albersheim. The theory and design of chirp radars. Bell System Techn. Journ. of 1960, v. 39, No 4. ("Foreign radio electronics", 1961, No 1).

40. Ye. S. Levitin, E. T. Polyak. Minimizations in the presence of limitations. Jour. comp. math. and math. physics, 1966, Vol. 6, No 5.

41. I. M. Lachenko. Methods of formation and processing of complicated radar signals. Survey/coverage of foreign inventions. Transactions TsNIIPI, Moscow, 1968.

42. L. A. Lyusternik, V. I. Sobolyev. Elements/cells of functional analysis. Publishing house "science", 1965.

43. Landau H. J., Pollak H. O. Prolate spheroidal wave functions. Fourier analysis and uncertainty—II. Bell System Techn. Journ. 1961, v. 40, No 1.

44. Lerner R. M. Signals with uniform ambiguity functions. IRE National Conv. Record, pt. 4, 1958.

45. K. A. Meshkovskiy, N. Ye. Kirillov. Coding in communication equipment. Publishing house "connection/communication", 1966.

46. G. S. Mikhlin. Integral equations. State Technical Press, 1949.

47. Maas G. L. A simplified calculation for Dolph-Tchebyscheff arrays. Journ. of Appl. Phys. 1954, v. 25, No 1

48. N. T. Petrovich, M. K. Razmakhin. Communicating systems with the noise-like signals. Publishing house "Soviet radio", 1969.

49. R. E. Millet. A matched-filter pulse-compression system using a nonlinear FM waveform. IEEE Trans. of 1970, v. AES-6, No-1. (Foreign radio electronics", 1970, No 9).

50. M. I. Pelakhatyy. On the sequences of quadratic residue with the best autocorrelation properties. "Radio engineering and electronics", 1971, Vol. 16, No 5.

51. U. Peterson. Codes, which correct errors. Publishing house "Mir", 1964.

52. B. T. Polyak. Minimizations of the functions of many variable/alternating. The "eccncy and math. methods", 1967, Vol. 3, No 6.

53. A. S. Polyanskiy, R. M. Sedletskiy. Synthesis of signals according to the assigned multipeak correlation function. "Radio engineering and electronics", 1969, Vol. 14, No 8.

54. Phillips C. S. E. Computer — controlled adaptive radar Electronics Record, 1967, v. 114, № 4.

55. Price R., Hoistetter E. M. Bounds on the volume and height distributions of the ambiguity functions. IEEE Trans. 1965, v. IT-11, № 2.

56. Ramp H. O., Wingrove E. R. Principle of pulse compression. IRE Trans. 1961, v. MIL-5, № 2.

57. Rihaczek A. Doppler — tolerant signal waveforms. Proc. IEEE, 1966, v. 54, № 6.

58. Rihaczek A., Mitchell R. L. Radar wave-forms for suppression of extended clutter. IEEE Trans. 1967, v. AP-15, № 3.

("Foreign radio electronics", 1968, No. 2)

59. G. S. Sebestian. Processes of making decisions with pattern recognition. Publishing house "Technology", Kiev, 1965.

60. R. M. Sedletskiy. Numerical methods of the synthesis of signals on the modulus/module of the function of uncertainty/indeterminacy. "Radio engineering and electronics", 1970, Vol. 15, No 4.

61. T. N. Sergeyenko. Synthesis of FM signal according to the autocorrelation function. "Radio engineering and electronics", 1970,

Vol. 15, No 3.

62. . K. Sloka. Questions of processing radar signals.

Publishing house "Soviet radic", 1970.

63. I. P. Sokolov, D. Ye. Vakman. Optimum linear broadside antenna arrays with the continuous current distribution. "Radio engineering and electronics", 1958, Vol. 3, No 1.

64. Schweppe F. C., Gray D. Z. Radar signal design subject to simultaneous peak and average power constraints. IEEE Trans. 1966, IT-12, No 1.

65. Slepian D. Some asymptotic expansions for prolate spheroidal wave functions. Journ. of Math. and Phys. 1965, v. 44, No 2.

66. Slepian D. Prolate spheroidal wave functions. Fourier analysis and uncertainty — IV. Bell System Techn. Journ. 1961, v. 40, No 3.

67. Slepian D., Pollak H. O. Prolate spheroidal wave function. Fourier analysis and uncertainty — I. Bell System Techn. Journ. 1961, v. 40, No 1.

68. Slepian D., Sonnenblick E. Eigenvalues associated with prolate spheroidal wave functions of zero order. Bell System Techn. Journ. 1965, v. 44, No 4.

69. L.G. Spafford. Optimum radar signal processing in clutter. IEEE Trans. of 1968, v. IT-14, No 5 ("Foreign radio electronics", 1969, No 10).

70. Stutt C. A. Some results on real-part, imaginary-part, and magnitude-phase relations in ambiguity functions. IEEE Trans. 1964, v. IT-10, No 4.

71. Stutt C. A., Spafford Z. J. A best-unmatched filter response for radar clutter discrimination. IEEE Trans. 1968, v. IT-14, No 2.

72. Sussman S. M. Least square synthesis of radar ambiguity functions. IRE Trans. 1962, v. IT-8, No 3.

73. L. B. Tartakovskiy. To the theory of the mirror of double curvature. "radio engineering and electronics", 1959, Vol. 4, No 11.

74. L. B. Tartakovskiy, V. K. Tikhonova. Synthesis of linear irradiator with assigned amplitude distribution. "Radio engineering and electronics", 1959, Vol. 4, No 12.

75. C. L. Tanes. Side-lobe suppression in range channel pulse-compression radar. IRE Trans. of 1962, v. MIL-6, No 2. ("Foreign radio electronics", 1963, No 5).

76. Titlebaum E. L. A generalization of a two dimensional Fourier transform property for ambiguity functions. IEEE Trans. 1966, v. IT-12, No 1.

77. Turin R. Storer J. On binary sequences. Proc. Am. Math. Soc. 1961, v. 12, No 2.

78. D. J. Wild. Methods of the search for extremum. Publishing house "science", 1967.

80. K. Flaumer. Table of spherical functions. Publishing house VTs of the AS USSR, 1962.

81. D. A. Haffmen. Synthesis of the linear multistage coding diagrams. In the collection "Theory of the transfer of communications/reports" edited by V. I. Siforov. Publishing house of foreign literature, 1957.

82. J. Khadli. Nonlinear and dynamic programming. Publishing house "Mir", 1967.

83. Ya. I. Khurgin, V. F. Yakovlev. Methods of the theory of integral functions in radiophysics, theory of connection/communication and optic/optics. Fizmatgiz, 1962.

84. N. Tsirler. Linear recurrent sequences. "Cybernetic collector/collection", iss. 7. Fizmatgiz, 1963.

85. Wolf J. D., Lee G. M., Sugi C. Radar waveforms synthesized by mean-square optimization techniques. IEEE Trans. AP-10, v. AES-5, No 4.

86. Ya. D. Shirman. A method to increase resolution of radars and a device for its accomplishment. Author's certificated No 146803 on application No 461974/40 of 15 July 1956. Byulleten' izobreteniy No. 9, 1962.

87. Ya. D. Shirman et al. From the history of Soviet radio engineering. "Radio engineering", 1970, Vol. 25, No 2.

88. A. Erdeyi. Asymptotic expansions. Fizmatgiz, 1962.

89. V. D. Yakovlev. Synthesis of the optimum sounding signals and filters. Transactions of the scientific-technical conference of the "Problem of optimum filtration". Publishing house "Soviet radio", 1967.

90. J. Danskin. Theory of the maximin. Publishing house "Soviet radio", 1970.

91. V. T. Dolgoychuk, M. B. Sverdlik. Optimization of reciprocal function of ambiguity in the assigned region. "Proceedings of VUZ of the USSR, Radio electronics", 1970, Vol. 13, No 2, page 186.

92. L. Ye. Varakin. Filtration of passive jamming and the partial space of the body of uncertainty/indeterminancy. "Proceedings of VUZ of the USSR", radic electronics, 1971, Vol. 14, No 10, page 1183.

93. Ye. Titchmarsh. Introduction to the theory of Fourier integrals. State Technical Press, 1948.

94. D. Ya. Vakman. On the determination of the concepts of amplitude, phase and instantaneous signal frequency. "Radio engineering and electronics", 1972, Vol. 17, No 5, page 972.

95. M. K. Razmakhnin, V. P. Yakovlev. Functions with the double orthogonality in radio electronics and optic/optics. Collection of translations. Sovetskoye radio Izdatel stvo.

96. G. Cramer, M. Lidsbetter. Stationary random processes. Publishing house "Soviet radio", 1971.

97. A. M. Zayezdnyy et al. Processing signals on the base of the use of their structural properties. "Radio engineering", 1971, Vol. 26, No 9.

98. V. I. Tikhonov. Statistical radio engineering. Publishing house "Soviet radio", 1966.

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87. Ya. D. Shirman, et al. From the History of Soviet Radio engineering. "Radio tekhnika"